MAT327 Tutorial 5

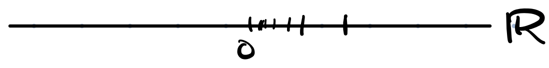
Post-Lecture Q2: identify the connected and path components of $\mathbb{Q}$, $\mathbb{R}_\ell$, $\mathbb{R}^{\mathbb{N}}$

($\mathbb{Q}$ as a subspace of $\mathbb{R}_{std}$; $\mathbb{R}_\ell$ generated by $[a,b)$; $\mathbb{R}^{\mathbb{N}}$ product)

Notation: for a pt x, C_x = connected component containing x,
P_x = path ——"—— .

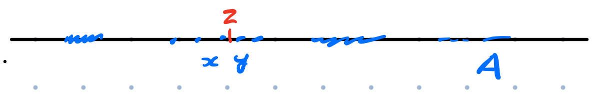
Always true: $\{x\} \subseteq P_x \subseteq C_x$.

$\mathbb{Q}$) For $x \in \mathbb{Q}$, $C_x = \{x\}$
$(\Rightarrow P_x = \{x\})$



Claim: if $A \subseteq \mathbb{Q}$ has $|A|>1$, then A is disconnected.

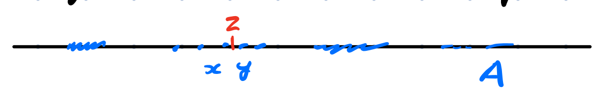
Pf. $|A|>1 \Rightarrow \exists\, x,y \in A$ s.t. $x<y$.
Choose z irrational s.t. $x<z<y$.
then $A = (A \cap (-\infty, z)) \cup (A \cap (z,\infty))$



Rmk. Spaces in which $C_x = \{x\}$ $\forall x$ are called *totally disconnected*.

Claim: $\mathbb{R}_\ell$ is totally disconnected.

Pf. Assume $A \subseteq \mathbb{R}_\ell$ w/ $|A|>1$. Choose $x,y \in A$ w/ $x<y$. Choose $z \in \mathbb{R}$ w/ $x<z<y$.
Then $A = (A \cap (-\infty, z)) \cup (A \cap [z,\infty))$

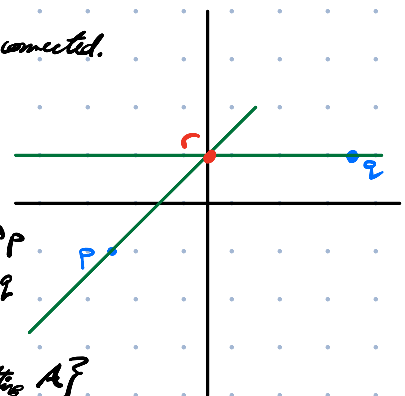


(Rmk. $(a,b) = \bigcup_{n\in\mathbb{N}} [a+\frac{1}{n}, b)$
$\Rightarrow \mathbb{R}_\ell$ is finer than $\mathbb{R}_{std}$)

$\mathbb{R}^{\mathbb{N}}$ has $C_x = P_x = \mathbb{R}^{\mathbb{N}}$ b/c $\mathbb{R}$ is (path-) connected and (path-)connectedness is arbitrarily productive.

Post-Lecture Q6: given any $A \subseteq \mathbb{R}^2$ countable, show that $\mathbb{R}^2 \setminus A$ is path-connected.

(Rmk. This implies that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected)



Pf. Fix $p,q \in \mathbb{R}^2 \setminus A$, want to construct $[0,1] \to \mathbb{R}^2 \setminus A$ mapping $0 \mapsto p$, $1 \mapsto q$.
Consider the set of all straight lines passing through p.
This is an uncountable set. Since A is ctable, at most ctably many of these lines hit A. $\Rightarrow$ {straight lines passing through p and not hitting A} is uncountable. Same argument $\Rightarrow$ {straight lines passing through q and not hitting A} is also uncountable.

Pick a line from each set. Can assume they are <u>not</u> parallel b/c (↑) is uncountable.
Then they have a unique intersection pt r. Follow the first line from p to r, then the second line from r to q

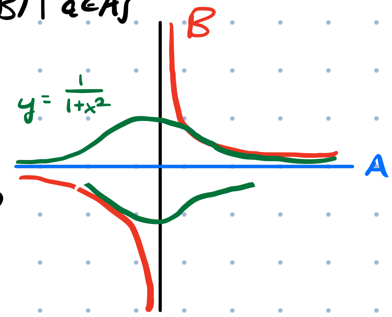
Revisit A2 Q1a

(X,d) metric space $\longrightarrow$ $d(x,B) := \inf\{d(x,b) \mid b \in B\}$

$d(A,B) := \inf\{d(a,b) \mid a\in A, b \in B\} = \inf\{d(a,B) \mid a \in A\}$

Q1a: if $A\cap B \neq \emptyset$, then $d(A,B)=0$.

Converse is false: NOT the case that $A \cap B = \emptyset \Rightarrow d(A,B)>0$, even if A and B are closed.



Claim: if A is compact and B is closed, then $A\cap B = \emptyset \Rightarrow d(A,B)>0$

Pf. Consider $d(\cdot,B): X \to [0,\infty)$. Claim: $d(\cdot, B)$ is a cts fn.

Restrict it to $d(\cdot,B): A \to [0,\infty)$. A is compact $\Rightarrow d(\cdot,B): A\to[0,\infty)$ achieves its min, say at $a\in A$. If $d(a,B)>0$, then we're done.

Recall Q1b: B is closed $\Leftrightarrow d(x,B)>0 \ \forall x \in B^c$. Since $A\cap B=\emptyset$, $a \in B^c \Rightarrow d(a,B)>0$

$d(a,B) = d(A,B)$ □

Compactness:

$\mathbb{R}_\ell$ $[a,b)$ ← NOT cpt $(\{[n,n+1)\})$

$\mathbb{R}_{cofinite}$ ← compact

$\mathbb{R}_{cocountable}$ ← NOT cpt

$\mathbb{R}_{ray}$ $(-\infty, a)$ ← NOT cpt

Pf. Start w/ any open cover $\mathcal{O}$. Pick $U\in\mathcal{O}$.

$\Rightarrow \mathbb{R}\setminus U = \{x_1,\dots,x_n\}$

$\mathcal{O}$ is an open cover $\Rightarrow$ each x_i occurs in some $U_i \in \mathcal{O}$

$\Rightarrow \{U, U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{O}$. □