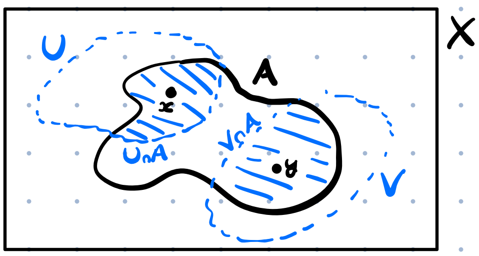
MAT327 Tutorial 4

So far, have encountered many properties of topological spaces:
- Hausdorff
- metrizable
- first countable
- second countable
- separable
- connected
- path-connected

Def. Let P be a property of topological spaces. We say P is
- *hereditary* if, whenever X is a space satisfying P, every subspace of X also satisfies P.
- (finitely/countably/arbitrarily) *productive* if, whenever $\{X_\alpha\}_{\alpha\in A}$ is a (finite/countable/arbitrary) collection of spaces all satisfying P, so does $\prod_{\alpha\in A} X_\alpha$.

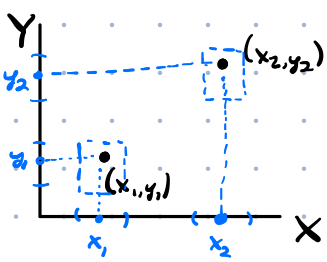
Property	Hereditary?	Finite prod.?	Countable prod.?	Arbitrary prod.?
Hausdorff	✓	✓	✓	✓
metrizable	✓	✓	✓	✗
first countable	✓	✓	✓	✗
second countable	✓	✓	✓	✗
separable	✗	✓	✓	✗
connected	✗	✓	✓	✓
path-connected	✗	✓	✓	✓

Hausdorffness is hereditary:

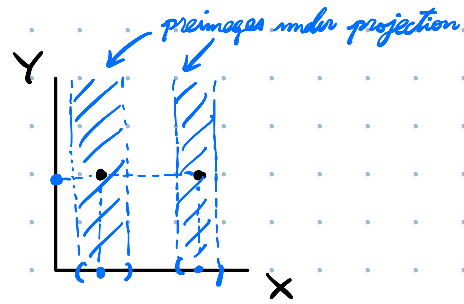


Hausdorffness is (arbitrarily) productive:

Idea does NOT work:



instead:



Metrizability is hereditary: $d: X\times X \to [0,\infty)$. Subspace topology of $A\subseteq X$ = metric topology of $d|_{A\times A}: A\times A\to[0,\infty)$.

First countability is hereditary: $\mathcal{N}_x$ a nbhd basis of $x\in A\subseteq X \Rightarrow \{U\cap A \mid U\in \mathcal{N}_x\}$ is a nbhd basis of $x\in A$.

Second countability is hereditary: $\mathcal{B}$ a basis of $X \Rightarrow \{U\cap A \mid U\in\mathcal{B}\}$ a basis of A.

Second countability is countably productive: $\{X_i\}_{i\in\mathbb{N}}$ 2nd ctable spaces $\rightsquigarrow$ $\mathcal{B}_i$ a ctable basis for X_i

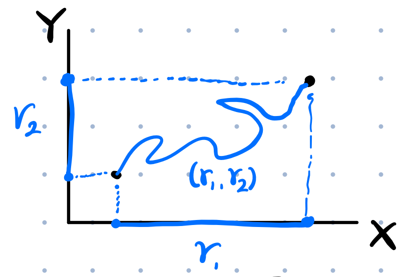
Naïve guess: $\{\prod_{i\in\mathbb{N}} U_i \mid U_i \in \mathcal{B}_i\}$ is not necessarily a basis of $\prod_{i\in\mathbb{N}} X_i$ (b/c $\prod_{i\in\mathbb{N}} U_i$ might be not open).

Instead: $\{\prod_{i\in\mathbb{N}} U_i \mid U_i = X_i$ for all but finitely many $i \in \mathbb{N}$, and otherwise $U_i \in \mathcal{B}_i\}$

Separability is <u>not</u> hereditary, but it is hereditary in the class of 2nd ctable spaces.
(b/c second countability is hereditary and implies separability)

Connectedness and path-connectedness are both not hereditary:

Path-connectedness is (arbitrarily) productive:



$\gamma_i : [0,1] \to (X \text{ or } Y)$

$\rightsquigarrow$

$(\gamma_1, \gamma_2) : [0,1] \to X \times Y$

$t \mapsto (\gamma_1(t), \gamma_2(t))$

For arbitrary products $\{X_\alpha\}_{\alpha\in A}$: $(x_\alpha)_{\alpha\in A}, (y_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha$

$\downarrow$

$x_\alpha, y_\alpha \in X_\alpha$

Connect these w/ $\gamma_\alpha : [0,1] \to X_\alpha$ $\rightsquigarrow$ $(\gamma_\alpha)_{\alpha\in A} : [0,1] \to X$

$\gamma_\alpha(0) = x_\alpha, \ \gamma_\alpha(1) = y_\alpha$

Connectedness is (arbitrarily) productive. Example:

Claim: $\mathbb{R}^{\mathbb{N}}$ is connected. (See Munkres p.151 ex. 7)

Idea: find a dense ($\bar{A} = \mathbb{R}^{\mathbb{N}}$) connected set $A \subset \mathbb{R}^{\mathbb{N}}$. Closure of connected set is connected $\Rightarrow \mathbb{R}^{\mathbb{N}}$ is connected.

Munkres takes $A = \mathbb{R}^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}^n$ (sequences that eventually vanish)

Alternative proof that $\bar{A} = \mathbb{R}^{\mathbb{N}}$ (uses 1st countability): fix $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, consider

$(x_1, 0, 0, 0, \dots)$
$(x_1, x_2, 0, 0, \dots)$
$(x_1, x_2, x_3, 0, \dots)$
$\downarrow$
$(x_1, x_2, x_3, \dots)$