MAT327 Tutorial 3

Prop. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a collection of 1^{st} ctble spaces indexed by an uncountable set A. If $X := \prod_{\alpha \in A} X_\alpha$ is first countable (↖ w/ the product topology), then all but countably many of the τ_α's are trivial (i.e., $\tau_\alpha = \{\emptyset, X_\alpha\}$).

Corollary. An uncountable product of metric spaces is generally ~~not~~ metrizable!

Pf of prop. BWOC, assume that X is 1^{st} ctble & uncountably many of the τ_α's are non-trivial. (★)

For each $\tau_\alpha \neq \{\emptyset, X_\alpha\}$, $\exists U_\alpha$ open that is $\neq \emptyset, X_\alpha$. Choose $x_\alpha \in U_\alpha \subsetneq X_\alpha$.

For each $\tau_\alpha = \{\emptyset, X_\alpha\}$, choose $x_\alpha \in X_\alpha$ arbitrarily.

Set $x := (x_\alpha)_{\alpha \in A} \in X$.

X 1^{st} ctble $\Rightarrow \exists \{V_n \mid n \in \mathbb{N}\}$ a nbhd basis of x.

WLOG, can assume each V_n is a basis set of the product topology.

($\{O_i\}$ a nbhd basis of x. O_i''s open s.t. $x \in O_i' \subseteq O_i \Rightarrow \{O_i'\}$ also a nbhd basis)

$\Rightarrow \forall n \in \mathbb{N}$, $\pi_\alpha(V_n) = X_\alpha$ for all but fin. many $\alpha \in A$, where $\pi_\alpha : X \to X_\alpha$, $(x_\alpha)_{\alpha \in A} \mapsto x_\alpha$.

i.e. $\forall n \in \mathbb{N}$, $F_n := \{\alpha \in A \mid \pi_\alpha(V_n) \neq X_\alpha\}$ is a finite set.

$\Rightarrow S = \bigcup_{n \in \mathbb{N}} F_n \ (\subseteq A)$ is countable.

What is $A \setminus S$? Exercise: $A \setminus S = \{\alpha \in A \mid \pi_\alpha(V_n) = X_\alpha \ \forall n \in \mathbb{N}\}$.

$A' = \{\alpha \in A \mid \tau_\alpha$ is non-trivial$\}$ is uncountable by assumption (★)

$\Rightarrow A' \setminus S = \{\alpha \in A \mid \pi_\alpha(V_n) = X_\alpha \ \forall n \in \mathbb{N}$ AND $\tau_\alpha \neq \{\emptyset, X_\alpha\}\}$ is nonempty (b/c uncountable set\countable = uncountable set)

Choose $\beta \to x_\beta \in U_\beta \overset{open}{\subsetneq} X_\beta \to x \in \pi_\beta^{-1}(U_\beta) \overset{open}{\subsetneq} X$

Claim: $\nexists n \in \mathbb{N}$ s.t. $V_n \subseteq \pi_\beta^{-1}(U_\beta)$. If there were such an n, then

$$X_\beta = \pi_\beta(V_n) \subseteq \pi_\beta(\pi_\beta^{-1}(U_\beta)) = U_\beta \Rightarrow X_\beta \subseteq U_\beta,$$ which contradicts $U_\beta \subsetneq X_\beta$.

(↑ b/c π_β surjective)

Thus $\{V_n\}$ is not actually a nbhd basis. □

(Exercise: rewrite above proof as a clean direct proof, i.e. without using proof by contradiction.)

*3. For each of the following functions $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, determine whether they are continuous or open when $\mathbb{R}^{\mathbb{N}}$ has the box, uniform, and product topologies. (One of these may be a bit trickier than the usual one-star problem.)

$f_1^{-1}\left(\prod_{n\in\mathbb{N}}(-\tfrac{1}{n}, \tfrac{1}{n})\right) = \{0\}$

$f_2^{-1}\left(\prod_{n\in\mathbb{N}}(-1, 1)\right)$ is not open

$f_3^{-1}\left(\prod_{n\in\mathbb{N}}(-\tfrac{1}{n^2}, \tfrac{1}{n^2})\right)$ is not open

Continuous?	τ_{prod} $\subseteq$	τ_{unif} $\subseteq$	τ_{box}
$f_1(t) = (t, t, t, t, t, \dots)$	Y		N
$f_2(t) = (t, 2t, 3t, 4t, 5t, \dots)$	Y		N
$f_3(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \frac{1}{4}t, \frac{1}{5}t, \dots)$	Y		N
$f_4(t) = (t, t^2, t^3, t^4, t^5, \dots)$	Y		N?
$f_5(t) = (t, \sqrt{t}, \sqrt[3]{t}, \sqrt[4]{t}, \sqrt[5]{t}, \dots)$	Y		N?

Open? Think of $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^2$, which has $f((-1,1)) = [0, 1)$. (↗ open; ↖ not open)

(Assume f_5 is defined on $[0, \infty)$.)

*2. For each of the following sequences in $\mathbb{R}^{\mathbb{N}}$, determine whether they converge in $\mathbb{R}^{\mathbb{N}}_{\text{box}}$, $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$, and $\mathbb{R}^{\mathbb{N}}_{\text{prod}}$.

$$\begin{aligned} a_1 &= (1,0,0,0,0,\dots) \\ a_2 &= (\tfrac{1}{2},\tfrac{1}{2},0,0,0,\dots) \\ a_3 &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,0,\dots) \\ a_4 &= (\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},0,\dots) \\ &\vdots \end{aligned} \qquad \begin{aligned} b_1 &= (1,1,1,1,1,\dots) \\ b_2 &= (0,2,2,2,2,\dots) \\ b_3 &= (0,0,3,3,3,\dots) \\ b_4 &= (0,0,0,4,4,\dots) \\ &\vdots \end{aligned}$$

$$\begin{aligned} c_1 &= (1,1,1,1,1,\dots) \\ c_2 &= (0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\dots) \\ c_3 &= (0,0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\dots) \\ c_4 &= (0,0,0,\tfrac{1}{4},\tfrac{1}{4},\dots) \\ &\vdots \end{aligned} \qquad \begin{aligned} d_1 &= (1,1,1,1,1,\dots) \\ d_2 &= (0,1,1,1,1,\dots) \\ d_3 &= (0,0,1,1,1,\dots) \\ d_4 &= (0,0,0,1,1,\dots) \\ &\vdots \end{aligned}$$

$$\begin{aligned} e_1 &= (1,0,0,0,0,\dots) \\ e_2 &= (0,1,0,0,0,\dots) \\ e_3 &= (0,0,1,0,0,\dots) \\ e_4 &= (0,0,0,1,0,\dots) \\ &\vdots \end{aligned} \qquad \begin{aligned} f_1 &= (1,0,0,0,0,\dots) \\ f_2 &= (0,\tfrac{1}{2},0,0,0,\dots) \\ f_3 &= (0,0,\tfrac{1}{3},0,0,\dots) \\ f_4 &= (0,0,0,\tfrac{1}{4},0,\dots) \\ &\vdots \end{aligned}$$

$\star$1. The purpose of this exercise is to work on your intuition about $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$. Recall that the uniform metric d_u on $\mathbb{R}^{\mathbb{N}}$ is defined by

$$d_u(x,y) = \sup\left\{\,\overline{d}(x_n,y_n)\;:\;n\in\mathbb{N}\,\right\}.$$

In the notes we said that an ϵ-ball around x in this metric is sort of like a "tube" around x, but this is not *quite* true. Fix $x \in \mathbb{R}^{\mathbb{N}}$ and $\epsilon > 0$, and define a subset:

$$\begin{aligned} U(x,\epsilon) &= \prod_{n\in\mathbb{N}} (x_n-\epsilon, x_n+\epsilon) \\ &= (x_1-\epsilon, x_1+\epsilon)\times(x_2-\epsilon,x_2+\epsilon)\times(x_3-\epsilon,x_3+\epsilon)\times\cdots \end{aligned}$$

This subset obviously contains x, and should be what you think of when we say "a tube around x". However:

(a) Show that $U(x,\epsilon) \neq B_\epsilon(x)$ (where $B_\epsilon(x)$ is the ϵ-ball around x according to d_u).

(b) Show that $U(x,\epsilon)$ is not open in $\mathbb{R}^{\mathbb{N}}_{\text{unif}}$.

(c) All is not lost, however. Show that for $\epsilon \leq 1$,

$$B_\epsilon(x) = \bigcup_{0<\delta<\epsilon} U(x,\delta).$$