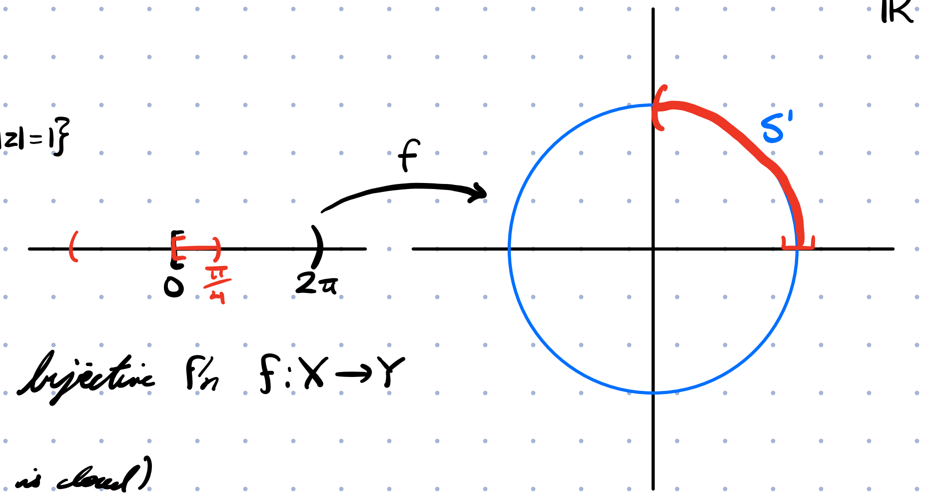
MAT327 Tutorial 2

S^1 as a subspace of $\mathbb{R}^2 \cong \mathbb{C}$

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \cong \{z \in \mathbb{C} \mid |z|=1\}$

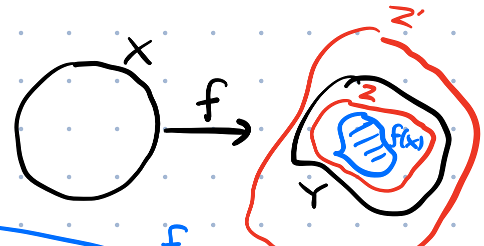


Subspace topology of $Y \subseteq (X,\tau)$:
$\tau_Y := \{U \cap Y \mid U \in \tau\}$

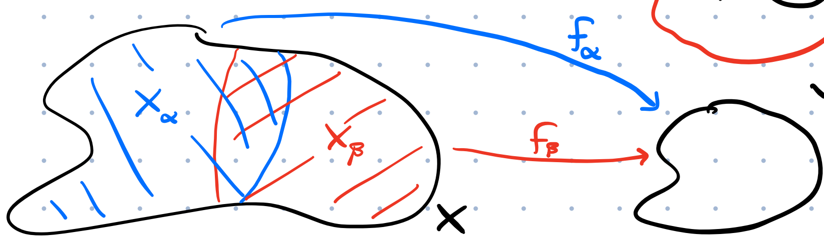
Recall: homeomorphism $X \to Y$ is a bijective fn $f: X \to Y$
s.t. f, f^{-1} are both cts.
($\Leftrightarrow f$ is open $\Leftrightarrow f$ is closed)

$f: [0, 2\pi) \longrightarrow S^1$, $f(\theta) = (\cos\theta, \sin\theta)$ is a bijection, cts, but f^{-1} is NOT continuous
($\Leftrightarrow f$ is NOT an open map)

Munkres Thm 18.2. Let X, Y, Z be topological spaces.
a) If $f: X \to Y$ is constant, then f is cts.
b) If $A \subseteq X$ is equipped w/ subspace topology, then $\iota: A \hookrightarrow X$ $\iota(x) = x$ (inclusion) is cts.
c) $f: X \to Y$ cts, $g: Y \to Z$ cts $\Rightarrow g \circ f: X \to Z$ is cts
d) If $f: X \to Y$ is cts and $A \subseteq X$, then $f|_A: A \to Y$ is cts.
e) If $f: X \to Y$ is cts and $f(X) \subseteq Z \subseteq Y$, then $g: X \to Z$, $g(x) = f(x)$ is cts. If $Y \subseteq Z'$, then $f: X \to Z'$ is cts.



b) For any $U \subseteq X$, $\iota^{-1}(U) = U \cap A$
c) $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$
d) $f|_A = f \circ \iota$
e) Exercise.



Part (f): Let $X = \bigcup_{\alpha \in I} X_\alpha$, $X_\alpha \subset X$ open. If for each $\alpha \in I$, we are given $f_\alpha: X_\alpha \to Y$.
If each f_α is cts, and $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$, then $\exists! \; f: X \to Y$ cts s.t. $f|_{X_\alpha} = f_\alpha$

Munkres Thm 18.3. Let X be a top space. Write $X = \bigcup_{i=1}^{N} X_i$ (finite!) where each $X_i \subseteq X$ is closed.
Given a top space Y and cts fns $f_i: X_i \to Y$ s.t. $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ $\forall i,j \in \{1, \dots, N\}$
there exists a unique cts fn $f: X \to Y$ s.t. $f|_{X_i} = f_i$.

$g: S^1 \to S^1$ uniquely determined by $S^1 = A \cup B$, where
$\begin{cases} A = \{\mathrm{Im}(z) \geq 0\}, & g_A: A \to S^1 \quad g_A(z) = z^2 \\ B = \{\mathrm{Im}(z) \leq 0\}, & g_B: B \to S^1 \quad g_B(z) = \bar{z}^2 \end{cases}$
closed sets

Q4 from post-lecture

a) X, Y top spaces, $f: X \to Y$ cts. Show: $\underbrace{\text{if } x_n \to x \text{ in } X, \text{ then } f(x_n) \to f(x) \text{ in } Y.}_{\text{"sequentially continuous"}}$

Pf. $x_n \to x \iff \forall U \ni x$ open, $\exists N > 0$ s.t. $x_n \in U$ $\forall n > N$.
$\iff \forall U \ni x$ open, the set $\{n \in \mathbb{N} \mid x_n \notin U\}$ is finite.

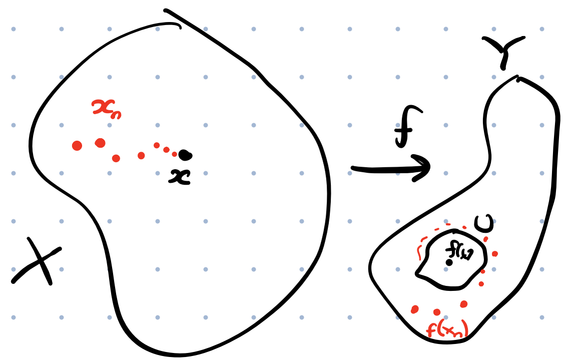
If $\overset{\subseteq Y}{U} \ni f(x)$ is open, $\{n \in \mathbb{N} \mid f(x_n) \notin U\} = \underbrace{\{n \in \mathbb{N} \mid x_n \notin f^{-1}(U)\}}_{\text{finite by assumption.}}$ $\square$

b) i) Show if X, Y are metric spaces, $f: X \to Y$ sequentially cts, then f is cts.
actually only need X to be a metric space

Pf sketch. • Given $U \subseteq Y$ open, WTS $f^{-1}(U) \subseteq X$ is open.
• i.e., $\forall x \in f^{-1}(U)$, $\exists r > 0$ s.t. $B_r(x) \subseteq f^{-1}(U)$ ← (uses basis)
• Claim: $\exists n \in \mathbb{N}$ s.t. $B_{1/n}(x) \subseteq f^{-1}(U)$. (So $r = \frac{1}{n}$ works).
• BWOC suppose not. Then

$$\forall n \in \mathbb{N}, \quad B_{1/n}(x) \nsubseteq f^{-1}(U)$$
$$\Updownarrow$$
$$B_{1/n}(x) \cap f^{-1}(U)^c \neq \emptyset$$

So I can choose $x_n \in B_{1/n}(x) \cap f^{-1}(U)^c$.
Exercise: show that $x_n \to x$.



$\square$