MAT 327 TUT # 11

Plan: - Thm 58.2 / 58.3 Munkres

- Deformation retraction & examples

Lemma: Let h, k be continuous maps $(X, \boxed{x_0}) \to (Y, \boxed{y_0})$

If h, k homotopic and if the $\boxed{\text{homotopy}}$ maps x_0 to y_0 for all $t \in [0,1]$,

then $\boxed{h_*}$ and $\boxed{k_*}$ are equal.

Recap: $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

$$h_*([f]) = [h \circ f]$$

Proof: WTS $\underline{h \circ f}$ and $\underline{k \circ f}$

(for any loop f based at x_0)

By assumption, $\exists$ H homotopy s.t.

$$H(x, 0) = h(x)$$
$$H(x, 1) = k(x)$$
$$H(x_0, t) = y_0 \quad \forall t \in [0,1]$$

$h \circ f \simeq k \circ f$

$\boxed{H(f(s), t)}$ $\forall (s, t) \in I \times I$

$H(f(s), 0) = h(x)$
$= h(f(s)) = h \circ f(s)$

$H(f(s), 1) = k \circ f(s)$

Path homotopy: based x_0

$H(\underbrace{f(0)}_{\substack{\parallel \\ x_0}}, t) = y_0$

$\blacksquare$

Thm 58.2.

The inclusion $\boxed{j : S^n \to \mathbb{R}^{n+1} \setminus \{0\}}$ induces an isomorphism of fundamental groups.

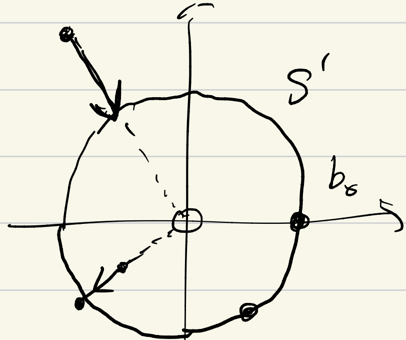
$(\pi_1(S^n, b_0) \underset{j_*}{\cong} \pi_1(\mathbb{R}^{n+1} \setminus \{0\}, b_0)$

Base point $b_0 \in S^n$

Proof: $X = \mathbb{R}^{n+1} \setminus \{0\}$

let $r : X \to S^n$

$$\boxed{r(x) = \frac{x}{\|x\|}}$$



j : inclusion

$r \circ j : X \to S^n$

is identity when restricted to S^n.

$$r \circ j \big|_{S^n} = id_{S^n}$$

$$\boxed{(r \circ j)_* = r_* \circ j_* = \text{identity}}$$
$$\Rightarrow \boxed{j_*} \text{ is injection}$$

Now $j \circ r$

$$X \xrightarrow{r} S^n \xrightarrow{j} X$$

We'll show $j \circ r$ homotopic to id_X

Straight line homotopy:

$$H(x, t) = (1-t) \underbrace{x}_{id} + t \frac{x}{\|x\|}$$

$t = 0$, $H(x, 0) = x = id$

$H(b_0, t) = b_0 \ \forall t$

because $b_0 \in S^n$

$\|b_0\| = 1$

Now, by lemma,

H is a homotopy that "preserves" the base point,

between id_X and $j \circ r$

So, $\boxed{id_{X*}} = (j \circ r)_* = \boxed{\boxed{j_*} \circ r_*}$

on $\pi_1(X, b_0)$

$\boxed{j_* \text{ surjective,}}$

We j_* $\boxed{\text{bijective}}$ homomorphism,

so isomorphism.

X and S^n have the same fundamental group. $\square$

Def $A \subset X$

A is a <u>deformation retract</u> of X

if $H : X \times I \to X$

$H(x, 0) = x$

$H(x, 1) \in A$

$H(a, t) = a \quad \forall a \in A \quad \forall t \in [0, 1]$

is called <u>deformation retraction</u>

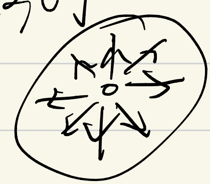
Thm 58.3.

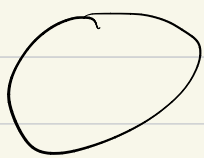
Let A be a deformation retract of X, and $x_0 \in A$

Then $j : (A, x_0) \to (X, x_0)$

induces j_* isomorphism

Example : $B \setminus \{0\}$ S^1





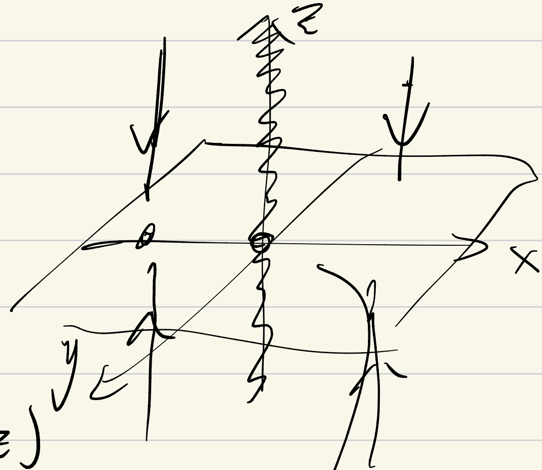
- $\mathbb{R}^3 \setminus \{z\text{-axis}\}$

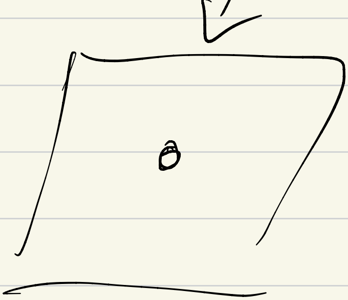
We can deform

$H(x, y, z, t)$

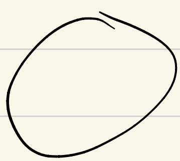
$= (x, y, (1-t)z)$

$\sim \mathbb{R}^2 \setminus \{0\}$





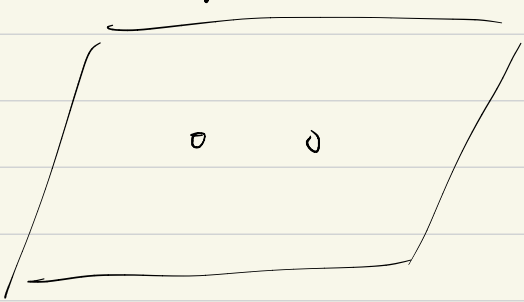
$\cong$



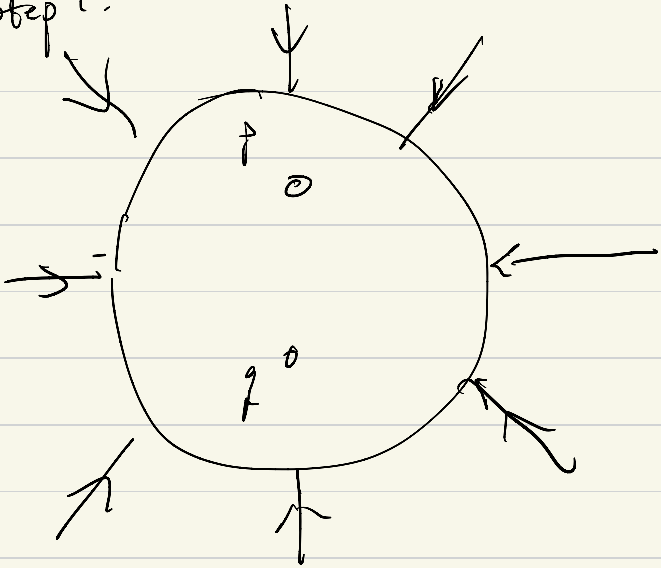
$S^1 \cong \mathbb{Z}$

Another example,

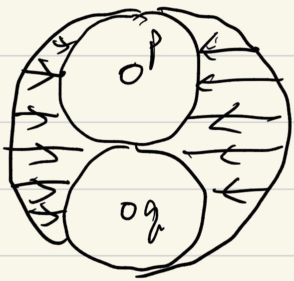
- double punctured plane: $\mathbb{R}^2 - p - q$



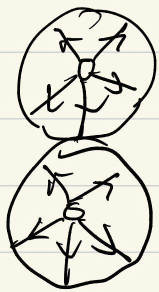
Step 1:



Step 2:

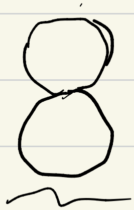


Step 3:



Resulting space:

$\mathbb{R}^2 - p - q$

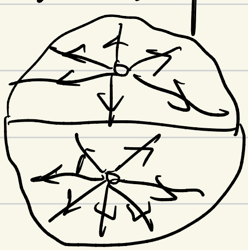


They are not deform. retracts of each other.

But they are

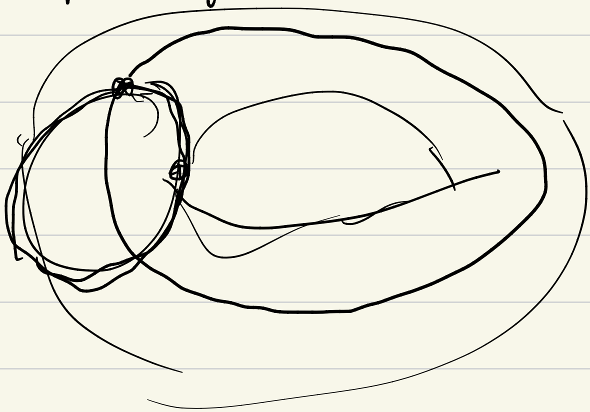
deform retracts of the same and hence have the same π_1.

Alternatively: at step 2, we can



More on "figure 8" space:

We can consider it as the subspace of torus



$p: \mathbb{R} \to S^1$

$p \times p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$

By restricting $\boxed{p \times p}$ to a preimage of figure 8.

We have a covering map of 8

$\mathbb{Z} \times \mathbb{Z}$