MAT 327 TUT # 10

Plan: Basic group theory (Appendix C from "topological manifold")

- definitions
- cosets and quotients
- 1st isomorphism theorem

Recap:

Def: Group $(G, \cdot)$,

a set with a binary operation $G \times G \to G$

We write $g \cdot h = gh$

1. Associativity $(gh)k = g(hk)$
2. Identity $\exists e$, s.t. $ge = eg = g$
3. Inverse For g, $\exists g^{-1}$, s.t. $gg^{-1} = g^{-1}g = e$

Order: $|G|$

Abelian group: "Commutative" $g, h \in G$ $gh = hg$

Subgroup:

$K \subset G$, K is closed under products and inverses we say K is a subgroup.

Now, consider maps that preserves structure

Def f is a homomorphism if

$f: (G_1, *_1) \to (G_2, *_2)$

$f(g *_1 h) = f(g) *_2 f(h)$

(simply write $f(gh) = f(g)f(h)$)

kernel of f: $\ker f = f^{-1}(e)$

Example: C_g, $h \mapsto ghg^{-1}$

$$ghkg^{-1} = gh(g^{-1}g)kg^{-1} = C_g(h)\,C_g(k)$$

Ex.

1. Identity is unique
2. Inverse is unique
3. Cancellation law make sense

   i.e. $au = av \Rightarrow u = v$
4. f injective $\Leftrightarrow$ $\ker f = \{e\}$
5. If f bijective then f^{-1} is also homomorphism.

   (Hence, if f is a bijective homomorphism, we say f is an isomorphism

   $f: G \to H$, G, H are isomorphic)
6. The image and preimage of a subgroup is still a subgroup.

   In particular, $\ker f = f^{-1}(\{e\})$ is also a subgroup.

__Def__: Subgroup $H \subset G$, $g \in G$

the (left) __coset__ of H by g is

$$gH = \{gh : h \in H\}$$

Congruence modulo H

$$g \equiv g' \pmod{H} \text{ iff } g^{-1}g' \in H$$

defines

a ...

an equivalence relation

Or in other words,

$$\exists \; h \in H, \text{ s.t. } hg = g'$$

Consider quotient

$$\boxed{G/H}$$

the elements are cosets gH

Def: $K \subset G$ is normal if $gKg^{-1} = K \quad \forall g \in G$

$$gKg^{-1} = K \iff gK = Kg$$

Def: For normal $K \subset G$, we have

$$(gK)(g'K) = (gg')K$$

on G/K

and G/K with this operation is a group.

Lemma: K normal in G

Given $f: G \to H$ homomorphism

and $\boxed{\ker f \supset K}$.

$\exists!$ $\tilde{f} : G/K \to H$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/K \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

and $f = \tilde{f} \circ \pi$.

(We say f descends to the quotient)

Proof: f is constant on gK

$$\tilde{f}(gK) = f(g)$$

The rest follows from homework.

<u>Thm</u>: (First Isomorphism Thm)

When $\boxed{K = \ker f}$. $f : G \to H$

f is surjective,

then $\tilde{f} : G/K \to H$

Proof: f surj $\Rightarrow$ $\tilde{f}$ surj

Recall: $\tilde{f}$ is inj. $\Leftrightarrow$ $\ker \tilde{f}$ is $\{e\}$

$$\ker f \Rightarrow \ker \hat{f} = \{eK\}$$

Hence, ______ $\square$

$$G/K = H$$

Note that $\ker f$ of an homomorphism is normal.

<u>Proof</u>: $gKg^{-1} = K$

Consider $h \in \ker f$,

$$f(h) = e$$

$$f(ghg^{-1}) = f(g)\, e\, \underline{f(g^{-1})}$$

$$= f(g)\, f(g)^{-1} = e$$

$\ker f$

f surjective,

$$G/\ker f = H$$

Cyclic Groups

Def A group G is cyclic if it is generated by a single element g.
i.e. elements are of the form $\boxed{g^n}$, $n \in \mathbb{Z}$.

Ex. a) $(\mathbb{Z}, +)$ is cyclic.
(not "e", e is 0)

b) $\mathbb{Z}/(n\mathbb{Z})$ has order n

integer powers of n

$g \equiv g' \pmod{n}$

$7 \equiv 1 \pmod{2}$

group under $+$

$\mathbb{Z}/2\mathbb{Z}$: $0, 1$

$\mathbb{Z}/7\mathbb{Z}$: $0, 1, \ldots, 6$

Suppose $K \subset G$, H

$f : G \to H$

K is closed under products / inverses

Claim: $\forall x, y \in K$,
$xy^{-1} \in K$, then we say K is subgroup

$f(K)$

$$
\begin{aligned}
& f(x)f(y)^{-1} \\
&= f(x)f(y^{-1}) \\
&= f(\underbrace{xy^{-1}}_{K}) \\
&\in f(K)
\end{aligned}
$$

Preimage: $K \subset H$

$$f^{-1}(K) \subset G$$

$$x, y \in f^{-1}(K)$$

$$f(x), f(y) \in K$$

$$f(xy^{-1}) = \boxed{f(x)\, f(y)^{-1}} \in K$$

$$xy^{-1} \in f^{-1}(K)$$

$\square$