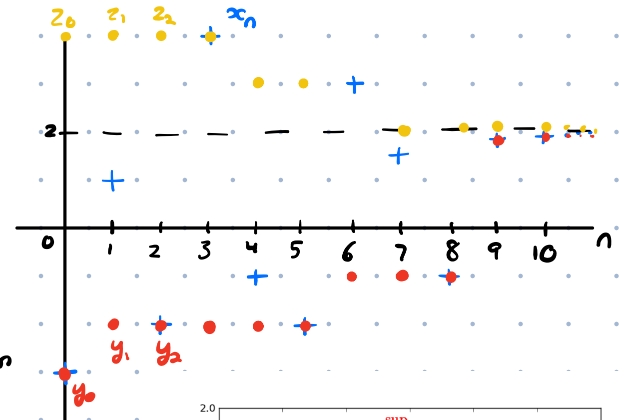
MAT327 Tutorial 1

- Introduction
- Limit superior & limit inferior: definition, intuition, examples, hint for PS1 (4a)
- Q3 from Friday post-lecture q's
- ~~(Time-permitting) Examples of closure, interior, limit pts in $\mathbb{R}$~~

Lim inf and lim sup

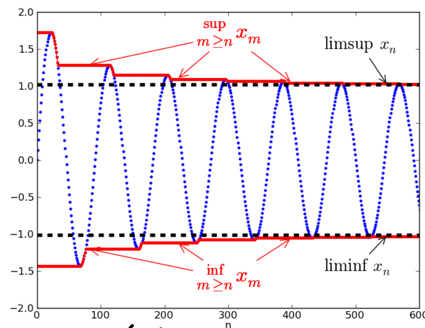


Def. Given a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$,

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right),$$

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right).$$

For each $n\in\mathbb{N}$, define $y_n := \inf_{m\geq n} x_m$. Why does $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ exist (or $=\infty$)? Exercise: $y_0 \leq y_1 \leq y_2 \leq \cdots$

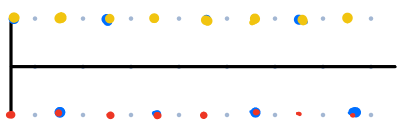


(from Wikipedia)

Similarly, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, where $z_n := \sup_{m\geq n} x_m$.

Example: $x_n = (-1)^n$. $\limsup_{n\to\infty} x_n = 1$

$\liminf_{n\to\infty} x_n = -1$



Proposition. $\lim_{n\to\infty} x_n$ exists $\iff \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ ($\leq$ always)

(i.e. x_n converges to x in standard topology of $\mathbb{R}$)

Pf sketch. $\forall n \in \mathbb{N}$, $y_n \leq x_n \leq z_n$. Apply squeeze theorem. $\square$

Alternative characterization of lim sup/inf:

- $\alpha \geq \sup_{n\in\mathbb{N}} x_n \iff \forall \varepsilon>0, \forall n\in\mathbb{N}, x_n < \alpha+\varepsilon$
- $\beta \leq \inf_{n\in\mathbb{N}} x_n \iff \forall \varepsilon>0, \forall n\in\mathbb{N}, x_n > \beta-\varepsilon$
- $\alpha \geq \limsup_{n\to\infty} x_n \iff \forall\varepsilon>0, \exists N\in\mathbb{N}$ s.t. $x_n<\alpha+\varepsilon$ for all $n>N$
- $\beta \leq \liminf_{n\to\infty} x_n \iff \forall\varepsilon>0, \exists N\in\mathbb{N}$ s.t. $x_n>\beta-\varepsilon$ for all $n>N$ ← useful for Q4(a)

Q4a: $x_n \longrightarrow x$ in ray topology $\iff x \leq \liminf_{n\to\infty} x_n$

Q3 from last Friday

Std metric on $\mathbb{R}^n$: $d_{Eu}: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ $\quad d_{Eu}(x,y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \|x - y\|$

"L^∞ metric": $d_{sq}: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ $\quad d_{sq}(x,y) := \max_{i=1,\dots,n} |x_i - y_i|$.

d_{Eu} metric $\longrightarrow$ basis of sets of the form $B_r(x) = \{y \in \mathbb{R}^n \mid d_{Eu}(x,y) < r\}$ $\rightsquigarrow$ topology τ_{Eu}
d_{sq} metric $\longrightarrow$ basis of sets of the form $S_r(x) = \{y \in \mathbb{R}^n \mid d_{sq}(x,y) < r\}$ $\rightsquigarrow$ topology τ_{sq}

Goal: show that $\tau_{Eu} = \tau_{sq}$.

$$\Updownarrow$$

$$\begin{cases} \tau_{Eu} \subseteq \tau_{sq} \Leftrightarrow \tau_{Eu} \text{ is coarser} \Leftrightarrow \tau_{sq} \text{ is finer} \\ \tau_{sq} \subseteq \tau_{Eu} \Leftrightarrow \tau_{sq} \text{ —〃— } \Leftrightarrow \tau_{Eu} \text{ —〃— } \end{cases}$$

every set $U \subseteq \mathbb{R}^n$ that is open in the Euclidean (standard) topology is also open in the "square topology" (topology defined by d_{sq}).

$U \subseteq \mathbb{R}^2$

https://www.desmos.com/calculator/1sttlv9216