# MAT 327: Introduction to Topology
# Assignment #6
# Due on Sunday August 13, 2023 by 11:59 pm

**Note:** This assignment covers material from Week #1-#11.

## Problem 1

In this problem, we will prove the Baire category theorem and study its powerful consequences in analysis and topology.

**(a)** **Baire Category Theorem** Let $X$ be a complete metric space or a locally compact Hausdorff space. Show that every countable collection of dense open sets has a dense intersection. Equivalently, show that the union of every countable collection of closed nowhere dense sets has empty interior.

*Remark:* A subset of $X$ is nowhere dense if its closure has empty interior.

**(b)** Let $\mathcal{F}$ be a collection of continuous real-valued functions on a locally compact Hausdorff space or a complete metric space. Suppose that $\mathcal{F}$ is piece-wise bounded in the sense that for every $x \in X$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$. Show that $\mathcal{F}$ is uniformly bounded on some open set $U \subseteq X$, meaning that there exists $M > 0$ such that $\sup_{x \in U} |f(x)| \leq M$ for all $f \in \mathcal{F}$.

*Hint:* Consider $A_n := \{x \in X \mid |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$.

**(c)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for every $x \in \mathbb{R}$, $\lim_{n\to\infty} f(nx) = 0$. Show that $\lim_{x\to\infty} f(x) = 0$.

**(d)** Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $D$ be the set of all points in $\mathbb{R}$ at which $f$ is continuous. Show that $D$ cannot be a countable dense set.

*Hint:* Show that $D$ is a $G_\delta$ set.

**(e)** **(*bonus*)** Equip $C([0, 1])$ with the supremum metric making it a complete metric space. For each $n \in \mathbb{N}$, let $F_n$ be the subset of $(C([0, 1], d_{\text{sup}})$ consisting of functions for which there is a point $x_0 \in [0, 1]$ such that $|(f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$. Show that $F_n$ is a closed nowhere dense set. Using the Baire category theorem, conclude that the set of continuous no-where differentiable functions is dense in $C([0, 1])$.

## Problem 2

**(a)** Show that every second countable, locally compact Hausdorff space is metrizable. Conclude that every $n$-manifold is metrizable.

**(b)** For a topological space, a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets is called an exhaustion of $X$ by compact sets if the $\bigcup_n K_n = X$ and $K_n \subseteq \text{Int}K_{n+1}$ for every $n \in \mathbb{N}$. Show that every second countable, locally compact Hausdorff space, and in particular an $n$-manifold, admits an exhaustion by compact sets.

**(c)** Let $X$ be a second countable, locally compact Hausdorff space and let $\{U_\alpha\}_{\alpha\in J}$ be an open cover for $X$. Show that there exists an open cover $\{V_\alpha\}_{\alpha\in J}$ such that $\overline{V}_\alpha \subseteq U_\alpha$ and that for every $x \in X$, there exists a neighbourhood of $x$ that intersects $V_\alpha$ for finitely many $\alpha \in J$.

*Remark:* This is a generalized form of the shrinking argument used in the proof of the existence of partition of unity in lectures.
*Hint:* Fix an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ by compact sets and define $A_n := K_{n+1} \setminus \text{Int}K_n$ and $U_n := \text{Int}K_{n+2} \setminus K_{n-1}$.

**(d)** ***(bonus)*** Generalize our proof for the existence of a partition of unity to second countable, locally compact Hausdorff spaces and, in particular, to n-manifolds.

## Problem 3

Let $A$ be a subspace of a topological space $X$. We say $A$ is a retract of $X$ if there exists a continuous map $r : X \to A$ such that $r|_A = Id_A$. The map $r$ is called a retraction.

In this problem, you can use without proof that the fundamental group of $S^1$ is isomorphic to the group $(\mathbb{Z}, +)$. We will prove this fact in the next lecture.

**(a)** Show that if $X$ is Hausdorff and $A$ is a retract of $X$, then $A$ is closed.

**(b)** Suppose that $A$ is a retract of $X$, Show that for any $x_0 \in A$, the homomorphism $\iota_* : \pi(A, x_0) \to \pi(X, x_0)$ induced by the inclusion map $\iota : A \to X$ is injective and the homomorphism $r_* : \pi(X, x_0) \to \pi(A, x_0)$ induced by the retraction $r$ is surjective. Conclude that a retract of a simply connected space is simply connected.

**(c)** Show that for any $n \in \mathbb{N}$, $S^{n-1}$ is a retract of $\mathbb{R}^n \setminus \{0\}$. Conclude that $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

**(d)** Show that the torus $T = S^1 \times S^1$ is not simply connected by finding a retract that is homeomorphic to $S^1$.

**(e)** Let $n \geq 2$ and let $N$ be the north pole in $S^n$. Let $p \in S^n \setminus \{N\}$. Show that any loop based at $p$ is homotopic to one that doesn't contain $N$. Conclude that $S^n$ is simply connected.

*Hint:* Recall that $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$.

**(f)** ***(bonus)*** Define the figure eight $Y \subseteq \mathbb{R}^2$ to be the union of the circles of radius one and centres $(0, 1)$ and $(0, -1)$. Show that $Y$ is not simply connected.