MAT 327: Introduction to Topology
Assignment #5
Due on Monday July 31, 2023 by 11:59 pm

Note: This assignment covers material from Week #1-#9.

Problem 1

Let $X := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Define a basis $\mathcal{B}$ for a topology on X as follows:

$$\mathcal{B} := \{B_\epsilon((a, b)) \mid a \in \mathbb{R}, b > 0, 0 < \epsilon < b\} \bigcup \{B_\epsilon((a, \epsilon)) \cup \{(a, 0)\} \mid \epsilon > 0, a \in \mathbb{R}\}$$

Convince yourself that $\mathcal{B}$ is indeed a basis for a topology on X. The space X equipped with this topology is called the Moore plane.

(a) Show that the Moore plane is completely regular.

Let A be a closed set and $p \in A^c$. Let B be a basis set containing p. Define a function $f : X \to [0, 1]$ as follows. First, define $f(p) := 0$. If $x \in B^c$, define $f(x) := 1$. If $x \in B \setminus \{p\}$, then let L_x be the line segment starting from p, passing through x, and ending at the boundary of B; define $f(x) := \frac{d(p,x)}{r}$ where d is the euclidean distance and r is the euclidean length of L.

We show that f is continuous. Let $x \in X$ and $\delta > 0$. Let $I = (f(x) - \delta, f(x) + \delta)$. It suffices to show that there exist a basis neighbourhood U of x such that $f(U) \subseteq I$. We first observe that $\overline{B}$ is the closed ball. If $x \in \overline{B}^c$, then we can choose U to be a neighbourhood basis of x contained in $\overline{B}^c$ and so $f(U) = \{1\} = \{f(x)\} \subseteq I$. If $x \in \overline{B}$, suppose for simplicity $x \neq p$. Then we can choose U to be a neighbouhood basis of x with diameter equal $\delta r/2$ where r is the length of L_x and so $f(U) \subseteq I$. The case when $x = p$ is similar.

(b) Let Z be a topological space with a dense subset D and a closed discrete subset C such that $|P(D)| \leq |C|$. Show that Z is not normal.

Suppose Z is normal, D is a dense subset and C is a closed discrete set. Since C is discrete and closed, A and $C \setminus A$ are closed disjoint sets for every $A \subseteq C$. For each set $A \subseteq C$, we choose disjoint nieghbourhoods U_A and V_A of A and $C \setminus A$. We define the function $F : P(C) \to P(D)$ as follows. Define $F(\emptyset) := \emptyset$ and $F(C) := D$. Otherwise, define $F(A) := U_A \cap D$. We show that F is injective. Suppsoe $A_1, A_2 \subseteq C$ such that $A_1 \neq A_2$. Assume without loss of generality that $A_2 \setminus A_1 \neq \emptyset$. Then $U_{A_2 \setminus A_1} \cap U_{A_2} \neq \emptyset$ and $V_{A_2 \setminus A_1} \cap U_{A_1} \neq \emptyset$. Since D is dense, we also have that $U_{A_2 \setminus A_1} \cap F(A_2) \neq \emptyset$ and

$V_{A_2\setminus A_1} \cap F(A_1) \neq \emptyset$. Since $U_{A_2\setminus A_1}$ is disjoint from $V_{A_2\setminus A_1}$, it follows that $F(A_1) \neq F(A_2)$ and hence F is injective. This implies that $|C| < |P(C)| \leq |P(D)|$ and so $|C| < |P(D)|$ as needed.

(c) Use the above to show that the Moore plane is not normal.

Let $C := \{(x, y) \in X \mid y = 0\}$ and $D = \{(x, y) \in X \mid x, y \in \mathbb{Q}\}$. It's easy to see that C is closed and discrete and D is dense. Since $|C| = |\mathbb{R}| = |P(\mathbb{Q}^2)| = |P(D)|$, it follows by part (b) that the Moore plane is not normal.

Let $Y := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(0, -1)\}$. We define a collection $\mathcal{B}$ of subsets of Y as follows. If $(x, y) \in Y$ for $y > 0$, then $\{(x, y)\} \in \mathcal{B}$. For each $x \in \mathbb{R}$, let $M_x := \{(x, y) \mid 0 \leq y < 2\}$, let $N_x := \{(x + y, y) \mid 0 \leq y < 2\}$, and let P_x be a finite subset of $(M_x \cup N_x) \setminus \{(x, 0)\}$. Then for each $x \in \mathbb{R}$, $(M_x \cup N_x) \setminus P_x$ is in $\mathcal{B}$. Also, for each $n \in \mathbb{N}$, $\{(x, y) \in Y \mid x > n\} \cup \{(0, -1)\}$ is in $\mathcal{B}$.

Draw a picture of the sets in $\mathcal{B}$ to help you visualize it and convince yourself that $\mathcal{B}$ is indeed a basis for a topology on Y.

(d) Show that Y is regular.

Hint: Consider three cases, $y = -1$, $y = 0$, and $y > 0$.

Let $(x, y) \in Y$ and let U be a neighbourhood of (x, y). If $y > 0$, then $V := \{(x, y)\}$ is a neighbourhood of (x, y) such that $\overline{V} = V \subseteq U$. If $y = 0$, then there exists a finite set $P_x \subseteq M_x \cup N_x$ such that $(M_x \cup N_x) \setminus P_x \subseteq U$. Letting $V := (M_x \cup N_x) \setminus P_x$, it follows that V is a neighbourhood of (x, y) such that $\overline{V} = V \subseteq U$. If $y = -1$, then there exists $n \in \mathbb{N}$ such that $\{(x, 0) \mid x > n\} \cup \{(0, -1)\} \subseteq U$. Letting $V := \{(x, 0) \mid x > n + 1\} \cup \{(0, -1)\}$, it follows that V is a neighbourhood of (x, y) such that $\overline{V} = \{(x, 0) \mid x \geq n + 1\} \cup \{(0, -1)\} \subseteq U$. We conclude that Y is regular.

(e) Let $C := \{(x, 0) \mid x \leq 1$, which is a closed set. Let $f : Y \to [0, 1]$ be a continuous map such that $f(C) = \{0\}$. For $n \in \mathbb{N}$, let $A_n := f^{-1}(0) \cap \{(x, 0) \mid n - 1 \leq x \leq n\}$. Show that A_n is infinite for every $n \in \mathbb{N}$.

We will prove it by induction. Since $A_1 = \{(x, 0) \mid 0 \leq x \leq 1\}$, A_1 is infinite. Suppose that A_n is infinite for some $n \in \mathbb{N}$. Fix a countable set $D \subseteq A_n$ and fix $(q, 0) \in D$. We will first show that $N_q \setminus f^{-1}(0)$ is countable. We write $N_q \setminus f^{-1}(0) = \cup_{k\in\mathbb{N}} E_k$ where $E_k := N_q \setminus f^{-1}(-1/k, 1/k)$. Suppose E_k is infinite; then every neighbouhood of $(q, 0)$ will contain N_q except for finitely many points and, hence, in particular will intersect E_k. This implies that $(q, 0) \in \overline{E_k} = E_k$, but we know that $f(q, 0) = 0$ by virtue of the fact that $(q, 0) \in A_n$. It follows that E_k is finite for every $k \in \mathbb{N}$ and so $N_q \setminus f^{-1}(0)$ is countable.

Define the set $P_q := \{(x,0) \mid x \in \pi_1(N_q \setminus f^{-1}(0))\}$ and $P := \cup_{(q,0)\in D} P_q$. Then define the set $F := \{(x,0) \mid n \leq x \leq n+1\} \setminus P$. Since P_q is countable for each $(q,0) \in D$ and D is countable, it follows that P is countable and hence F is infinite. We will show that $F \subseteq f^{-1}(0)$. Let $(x,0) \in F$. Then every neighbourhood of $(x,0)$ must contain M_x except for finitely many points; in particular, by definition of F, M_x must intersect $N_q \cap f^{-1}(0)$ for some $(q,0) \in D$. It follows that $(x,0) \in \overline{f^{-1}(0)} = f^{-1}(0)$.

Since $F \subseteq f^{-1}(0)$ and $F \subseteq \{(x,0) \mid n \leq x \leq n+1\}$, it follows that $F \subseteq A_{n+1}$. Since F is infinite, we conclude that A_{n+1} is infinite.

(f) Use the above to show that Y is not completely regular.

Hint: Show first that $f(0,-1) = 0$.
Remark: In #11 in section 33, Munkres constructs another topological space that is regular but not completely regular.

Let C be the closed set from part (e) and let $f : X \to [0.1]$ be a continuous function such that $f(C) = \{0\}$. By part (e), we know that A_n is nonempty for every $n \in \mathbb{N}$. Then every basis neighbouhood of $(0,-1)$ will contain A_n for some $n \in \mathbb{N}$ and therefore intersect $f^{-1}(0)$. This implies that $(0,-1) \in \overline{f^{-1}(0)} = f^{-1}(0)$ and so $f(0,-1) = 0$. We conclude that that the closed set C and the point $(0,-1)$ cannot be separated by a continuous function and so Y is not completely regular.

Problem 2

We will define a new separation axiom.
We say that a normal topological space X is perfectly normal (T_6) if for every pair of disjoint closed sets A and B, there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

(a) We say a closed set $A \subseteq X$ is a G_δ set if it's the intersection of countably many open sets. Let X be a normal space and let A and B be disjoint closed sets. Show that A and B are G_δ sets if and only if there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

Remark: This is usually called the strong version of the Urysohn Lemma.

$\Longrightarrow$

Suppose A and B are G_δ sets; so $A = \cap_n U_n$ and $B = \cap_n V_n$ where U_n and V_n are open sets.

For each n, A and U_n^c as well as B and V_n^c are disjoint closed sets and so we can invoke Urysohn Lemma to choose continuous functions $h_n, g_n : X \to [0,1]$ such that

$$h_n(A) = 0, \quad h_n(U_n^c) = 1, \quad g_n(B) = 0, \quad g_n(V_n^c) = 1$$

Then we define the functions $h, g : X \to [0,1]$ as follows:

$$h(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} h_n(x), \quad g(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(x)$$

Since the sum converges uniformly on X, it follows that h and g are continuous. If $h(x) = 0$, then $h_n(x) = 0$ for every $n \in \mathbb{N}$ implying that $x \in U_n$ for every $n \in \mathbb{N}$. It follows that $x \in \cap_n U_n = A$ and so $h^{-1}(0) = A$. Similarly, we have that $g^{-1}(0) = B$. Then the function $h : X \to [0,1]$ defined by

$$f(x) := \frac{h(x)}{h(x) + g(x)}$$

is the desired function.

$\Longleftarrow$

Suppose there exists a function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. Then $A = \cap_n U_n$ and $B = \cap_n V_n$ where

$$U_n = f^{-1}(-1/n, 1/n), \quad V_n = f^{-1}(1 - 1/n, 1 + 1/n)$$

We conclude that A and B are G_δ sets as needed.

(b) Let X be a normal space. Show that X is perfectly normal if and only if every closed set is a G_δ set if and only if every closed set A is $f^{-1}(\{0\})$ for some continuous function $f : X \to [0,1]$.

Suppose X is perfectly normal. Let A be a closed set. If $A = X$, then A is trivially a G_δ set and can be written as $f^{-1}(0)$ where $f = 0$. Otherwise, let $B = \{x\}$ where $x \notin A$. By the definition of perfectly normal, there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. By the backward direction in part(a), it follows that A is a G_δ set. This shows that the first statement implies the second and third.

By part (a), the second statement trivially implies the first.

The proof of the forward direction in part (a) shows in particular that if two disjoint closed sets can be written as $A = h^{-1}(0)$ and $B = g^{-1}(0)$ for some continuous functions $h, g : X \to [0,1]$, then there exists a function $f : X \to [0,1]$ defined by $f := \frac{h}{h+g}$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. This shows that the third statement implies the first. We have then shown that all three statements are equivalent.

(c) Show that every metrizable space is perfectly normal.

Let A and B be disjoint closed sets. Since A and B are closed, $d(x, A) = 0$ iff $x \in A$ and similarly for B. Then $f : X \to [0, 1]$ defined by $f(x) := \frac{d(x,A)}{d(x,A)+d(x,B)}$ is the desired function.

(d) Show that perfectly normal is strictly stronger than completely normal.

To show that perfectly normal is stronger than completely normal, it suffices to show that the property of being perfectly normal is hereditary. If so, then a perfectly normal space is in particular a normal space that is hereditarily normal, and hence completely normal. Let Y be a subspace of a perfectly normal space X and let A and B be disjoint closed set in Y. Then $A = \tilde{A} \cap Y$ and $B = \tilde{B} \cap Y$ for some closed sets $\tilde{A}$ and $\tilde{B}$ in X. Since X is perfectly normal, there exist continuous functions $h, g : X \to [0, 1]$ such that $h^{-1}(0) = \tilde{A}$ and $g^{-1}(0) = \tilde{B}$. Then $h|_Y, g|_Y : Y \to [0, 1]$ are continuous functions such that $h|_Y{}^{-1}(0) = A$ and $g|_Y{}^{-1}(0) = B$. Define the function $f : Y \to [0, 1]$ by $f := \frac{h|_Y}{h|_Y + g|_Y}$; then f satisfies $f^{-1}(0) = A$ and $f^{-1}(1) = B$. In particular, $f^{-1}([0, 1/4)$ and $f^{-1}(3/4, 1]$ are disjoint open neighbourhoods of A and B, and hence Y is normal. The existence of such a function f for every pair of disjoint closed sets in Y implies that Y is perfectly normal as needed.

Consider $(\mathbb{R}, \mathcal{T})$ defined in the hint. We will show that this space is completely normal but nor perfectly normal. It is clear that $(\mathbb{R}, \mathcal{T})$ is a T_1 space since $\mathbb{R} \setminus \{p\}$ is open for every $p \in \mathbb{R}$. Let A and B be subsets of $\mathbb{R}$ such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. We want to show that there exists disjoint neighbouhoods of A and B. At least one of A or B does not contain 0. if $0 \notin A$ and $0 \in B$, $\overline{A} = A \cup \{0\}$ and the condition $\overline{A} \cap B = \emptyset$ cannot be satisfied. If both A and B do not contain 0, then they are both open sets. The statement then follows trivially since A and B are disjoint neighbouhoods of A and B, respectively. We conclude that $(\mathbb{R}, \mathcal{T})$ is completely normal.

We now show that X is not perfectly normal. Consider the set $\mathbb{Z}$, which is a closed set as it contains 0. Notice that any neighbourhood of $\mathbb{Z}$ must be of the form $\mathbb{R} \setminus P$ for some finite set P. If $\mathbb{Z}$ is a G_δ set, then it can be written as a countable intersection of neighbourhoods of $\mathbb{Z}$. However, such a set must be of the form $\mathbb{R} \setminus D$ for some countable or finite set D and hence uncountable. Since Z is countable, it cannot be a G_δ set. We conclude that X is not perfectly normal as there exists a closed set that is not a G_δ set.

Hint: Consider $(\mathbb{R}, \mathcal{T})$ where $\mathcal{T} := \{U \subseteq \mathbb{R} \mid 0 \notin U \text{ or } U^c \text{ is finite } \}$.

Problem 3 *(bonus)*

We have defined many separation axioms and have proven the following hierarchy:

$$T_6 \implies T_5 \implies T_4 \implies T_{3.5} \implies T_3 \implies T_2 \implies T_1$$

Find topological spaces X_i, where $i \in S := \{1, 2, 3, 3.5, 4, 5\}$, such that X_i is T_i but not T_j where j is the next number in S. You do not need to prove that these topological spaces satisfy what you claim they do.

- Let $X = \{x, y\}$ where $\{x\}$ is the only nontrivial open set. Then X is T_1 but not T_2.
- The space $\mathbb{R}_K$ defined in lectures is T_2 but not T_3.
- The space Y defined in problem 1 is T_3 but not $T_{3.5}$.
- The Moore plane defined in problem 1 is $T_{3.5}$ but not T_4.
- The space $[0, 1]^{\mathbb{R}}$ is T_4 but not T_5.
- The space $(\mathbb{R}, \mathcal{T})$ defined in problem 2d is T_5 but not T_6.