MAT 327: Introduction to Topology
Assignment #5
Due on Monday July 31, 2023 by 11:59 pm

Note: This assignment covers material from Week #1-#9.

Problem 1

Let $X := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Define a basis $\mathcal{B}$ for a topology on X as follows:

$$\mathcal{B} := \{B_\epsilon((a, b)) \mid a \in \mathbb{R}, b > 0, 0 < \epsilon < b\} \bigcup \{B_\epsilon((a, \epsilon)) \cup \{(a, 0)\} \mid \epsilon > 0, a \in \mathbb{R}\}$$

Convince yourself that $\mathcal{B}$ is indeed a basis for a topology on X. The space X equipped with this topology is called the Moore plane.

(a) Show that the Moore plane is completely regular.

(b) Let Z be a topological space with a dense subset D and a closed discrete subset C such that $|P(D)| \leq |C|$. Show that Z is not normal.

(c) Use the above to show that the Moore plane is not normal.

Let $Y := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(0, -1)\}$. We define a collection $\mathcal{B}$ of subsets of Y as follows. If $(x, y) \in Y$ for $y > 0$, then $\{(x, y)\} \in \mathcal{B}$. For each $x \in \mathbb{R}$, let $M_x := \{(x, y) \mid 0 \leq y < 2\}$, let $N_x := \{(x + y, y) \mid 0 \leq y < 2\}$, and let P_x be a finite subset of $(M_x \cup N_x) \setminus \{(x, 0)\}$. Then for each $x \in \mathbb{R}$, $(M_x \cup N_x) \setminus P_x$ is in $\mathcal{B}$. Also, for each $n \in \mathbb{N}$, $\{(x, y) \in Y \mid x > n\} \cup \{(0, -1)\}$ is in $\mathcal{B}$.

Draw a picture of the sets in $\mathcal{B}$ to help you visualize it and convince yourself that $\mathcal{B}$ is indeed a basis for a topology on Y.

(d) Show that Y is regular.

Hint: Consider three cases, $y = -1$, $y = 0$, and $y > 0$.

(e) Let $C := \{(x, 0) \mid x \leq 1$, which is a closed set. Let $f : Y \to [0, 1]$ be a continuous map such that $f(C) = \{0\}$. For $n \in \mathbb{N}$, let $A_n := f^{-1}(0) \cap \{(x, 0) \mid n - 1 \leq x \leq n\}$. Show that A_n is infinite for every $n \in \mathbb{N}$.

(f) Use the above to show that Y is not completely regular.

Hint: Show first that $f(0, -1) = 0$.
Remark: In #11 in section 33, Munkres constructs another topological space that is regular but not completely regular.

Problem 2

We will define a new separation axiom.
We say that a normal topological space X is perfectly normal (T_6) if for every pair of disjoint closed sets A and B, there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

(a) We say a closed set $A \subseteq X$ is a G_δ set if it's the intersection of countably many open sets. Let X be a normal space and let A and B be disjoint closed sets. Show that A and B are G_δ sets if and only if there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

Remark: This is usually called the strong version of the Urysohn Lemma.

(b) Let X be a normal space. Show that X is perfectly normal if and only if every closed set is a G_δ set if and only if every closed set A is $f^{-1}(\{0\})$ for some continuous function $f : X \to [0, 1]$.

(c) Show that every metrizable space is perfectly normal.

(d) Show that perfectly normal is strictly stronger than completely normal.

Hint: Consider $(\mathbb{R}, \mathcal{T})$ where $\mathcal{T} := \{U \subseteq \mathbb{R} \mid 0 \notin U \text{ or } U^c \text{ is finite } \}$.

Problem 3 *(bonus)*

We have defined many separation axioms and have proven the following hierarchy:

$$T_6 \implies T_5 \implies T_4 \implies T_{3.5} \implies T_3 \implies T_2 \implies T_1$$

Find topological spaces X_i, where $i \in S := \{1, 2, 3, 3.5, 4, 5\}$, such that X_i is T_i but not T_j where j is the next number in S. You do not need to prove that these topological spaces satisfy what you claim they do.

Problem 4 (optional)

Let X be a set. A nonempty collection $\mathcal{F} \subseteq P(X)$ is called a filter on X if the following are satisfied:

- $\emptyset \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an ultrafilter if it is not properly contained in any other filter on X.

(a) Use Zorn's lemma to prove that every filter on X is contained in an ultrafilter.

(b) Show that a filter $\mathcal{U} \subseteq P(X)$ is an ultrafilter if and only if for every $A \subseteq X$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$. Conclude that for every $a \in A$, $\mathcal{U}_a := \{A \subseteq X \mid a \in A\}$ is an ultrafilter.

Remark: $\mathcal{U}_a$ is called the principal ultrafilter at a.

Let $\mathcal{Y}$ be the collection of all ultrafilters on $\mathbb{N}$, and give it the topology generated by the basis $\mathcal{B} := \{\mathcal{B}_A \mid A \subseteq \mathbb{N}$, where $\mathcal{B}_A := \{\mathcal{U} \in \mathcal{Y} \mid A \in \mathcal{U}\}$. We will show that $\mathcal{B}$ is a basis for $\mathcal{Y}$ making it the Stone-Čech compactification of $\mathbb{N}$.

(c) Show that $\mathcal{B}$ is a basis for a topology on $\mathcal{Y}$.

(d) Define the map $i : \mathbb{N} \to \mathcal{Y}$ by $i(n) := \mathcal{U}_n$, where $\mathcal{U}_n$ is the principal ultrafilter at n. Show the following:

- i is an embedding.
- $i(\mathbb{N})$ is dense in $\mathcal{Y}$.
- $\mathcal{Y}$ is compact and Hausdorff.
- Every bounded function $f : i(\mathbb{N}) \to \mathbb{R}$ can be uniquely extended to a continuous function $\tilde{f} : \mathcal{Y} \to \mathbb{R}$.

Conclude that $\mathcal{Y}$ is the Stone-Čech compactification of $\mathbb{N}$.