Question 1

Problem 1(a)

Let X be a compact metric space and $f\colon X \to X$ an isometry. Show that f is a homeomorphism.

Solution

First, note that *any* isometry $f\colon X \to Y$ of metric spaces (which need not be compact) is continuous by the ε-δ characterization of continuity in metric spaces (with $\delta = \varepsilon$) and injective: if $x, y \in X$ are such that $f(x) = f(y)$, then

$$d_X(x, y) = d_Y(f(x), f(y)) = 0,$$

hence also $x = y$. Thus f is bijective onto its image, and in fact $f^{-1}\colon f(X) \to X$ is also an isometry (exercise), hence continuous. It follows that $f\colon X \to f(X)$ is a homeomorphism, where $f(X)$ is endowed with the subspace topology. Therefore, in the case where $Y = X$ is a compact metric space, we need only check that f is surjective to conclude that it is a homeomorphism.

To wit, fix $x \in X$. We recursively define a sequence by $x_0 = x$ and $x_n = f(x_{n-1}) = f^n(x_0)$, so that $x_n \in f(X)$ for all $n \geq 1$. Since X is compact, the sequence (x_n) has a convergent subsequence, which is in particular Cauchy. It follows that, for every $\varepsilon > 0$, there exist $m, n \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$. Without loss of generality, assume that $m < n$; then

$$d(x, f(X)) \leq d(x_0, x_{n-m}) = d(x_m, x_n) < \varepsilon,$$

where the middle equality uses the fact that f is an isometry. Since $d(x, f(X)) < \varepsilon$ for all $\varepsilon > 0$, we conclude that $d(x, f(X)) = 0$. Since $f(X)$ is compact, it follows that $x \in f(X)$ (see the solution to Problem Set 2 Q1(a)).

Problem 1(b)

Let X be a compact metric space and $f\colon X \to X$ be a shrinking map. Show that there exists a unique fixed point.

Solution

Existence: The function $g\colon X \to [0, \infty)$ given by $g(x) = d(x, f(x))$ is continuous, as it is a composition of continuous functions. Since X is compact, g achieves its minimum by the extreme value theorem, say at $x \in X$. We claim that $f(x) = x$. Indeed, if it were the case that $f(x) \neq x$, then since f is shrinking,

$$g(f(x)) = d(f(x), f(f(x))) < d(x, f(x)) = g(x),$$

which contradicts the fact that x minimizes g.

Uniqueness: If there existed distinct fixed points $x \neq y$ of f, then

$$d(x, y) = d(f(x), f(y)) < d(x, y),$$

which is impossible.

Problem 1(c)

Let X be a complete metric space and $f\colon X \to X$ be a contraction. Show that there exists a unique fixed point.

Solution

Existence: Choose any $x_0 \in X$ and let $x_n = f(x_{n-1}) = f^n(x_0)$ as in part (a). If $x_1 = x_0$, then we are done, so assume that $x_1 \neq x_0$.

Claim: The sequence (x_n) is Cauchy. From this, it follows that the sequence converges to a point $x \in X$, as X is complete. Then x is a fixed point of f, as

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\left(\lim_{n\to\infty} x_{n-1}\right) = f(x),$$

where we used continuity of f for the third equality. (A contraction is Lipschitz continuous, hence continuous.)

Proof of claim. First, note that for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_n) = d(f^n(x_1), f^n(x_0)) \leq \alpha^n d(x_1, x_0),$$

where $\alpha \in [0,1)$ is such that $d(f(x), f(y)) \leq \alpha d(x,y)$ for all $x, y \in X$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$\alpha^N < \frac{1-\alpha}{d(x_1, x_0)}\varepsilon.$$

(This is possible since $\alpha \in [0,1)$ and $d(x_1, x_0) > 0$ by assumption.) Given $m > n > N$, we estimate

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{k=1}^{m-n} d(x_{n+k}, x_{n+k-1}) && \text{(by the triangle inequality)} \\ &\leq \sum_{k=1}^{m-n} \alpha^{n+k-1} d(x_1, x_0) \\ &= \alpha^n d(x_1, x_0) \sum_{k=1}^{m-n} \alpha^{k-1} \\ &\leq \alpha^n d(x_1, x_0) \sum_{k=0}^{\infty} \alpha^k \\ &= \alpha^n d(x_1, x_0) \frac{1}{1-\alpha}. \end{aligned}$$

Since $n > N$ and $0 \leq \alpha < 1$, we have $\alpha^n < \alpha^N$, hence

$$d(x_m, x_n) < \frac{d(x_1, x_0)}{1-\alpha}\alpha^N < \frac{d(x_1, x_0)}{1-\alpha}\frac{1-\alpha}{d(x_1, x_0)}\varepsilon = \varepsilon.$$

■

Uniqueness: The same proof idea as in part (b) works; namely, if $x \neq y$ were both fixed points of f, then

$$d(x,y) = d(f(x), f(y)) \leq \alpha d(x,y) < d(x,y),$$

which is impossible.

Problem 1(d)

Show that if "compact" was replaced with "complete" in (b), the statement would be false. *Hint:* Find a shrinking map $f\colon \mathbb{R} \to \mathbb{R}$ that has no fixed point.

Solution

Following the hint, we define $f\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \ln(e^x + 1).$$

This is well-defined and smooth; moreover, $f'(x) = \frac{e^x}{e^x+1}$ satisfies $|f'| < 1$.
f is shrinking: Fix $x, y \in \mathbb{R}$ with $x > y$. Using the fundamental theorem of calculus and the fact that $|f'| < 1$, we compute

$$|f(x) - f(y)| = \left|\int_y^x f'(t)\,dt\right| \leq \int_y^x |f'(t)|\,dt < |x - y|.$$

f has no fixed point: For all $x \in \mathbb{R}$, we have $f(x) = \ln(e^x + 1) > \ln(e^x) = x$.

Problem 1(e) (bonus)

Would (a) be true if the codomain of f is a different compact metric space Y? What if we add the assumption that f is surjective?

Solution

Without the assumption that $f\colon X \to Y$ is surjective, the statement of part (a) is false. For instance, let Y be any compact metric space with at least two points, fix a point $x \in Y$, and set $X = \{x\}$. Then the inclusion $f\colon X \hookrightarrow Y$ is an isometry of compact metric spaces but is certainly not a homeomorphism, as it is not surjective.
However, (failure of) surjectivity is the *only* obstruction to an isometry of compact metric spaces being a homeomorphism. Indeed, this is what we saw in the solution to part (a).

Question 2

Problem 2(a)

Which of the four definitions of compactness is preserved by continuous functions? Which of the four definitions satisfy the extreme value theorem?

Solution

Compactness: This is preserved by continuous functions (Lecture 11) and satisfies the extreme value theorem (Lecture 12).
Sequential compactness: This is preserved by continuous functions: suppose X is sequentially compact and $f\colon X \to Y$ is continuous. Let $\{y_n\}_{n\in\mathbb{N}}$ be any sequence in $f(X)$. For

each $n \in \mathbb{N}$, choose $x_n \in X$ such that $y_n = f(x_n)$. Since X is sequentially compact, $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence, say $x_{n_k} \to x \in X$. Then $y_{n_k} = f(x_{n_k}) \to f(x)$.
This satisfies the extreme value theorem: if X is sequentially compact and $f\colon X \to \mathbb{R}$ is continuous, then by the above, $f(X)$ is a sequentially compact subset of $\mathbb{R}$. Since $\mathbb{R}$ is metrizable, this is equivalent to $f(X)$ being compact. The Heine–Borel theorem yields the conclusion.
Countable compactness: This is preserved by continuous functions: suppose X is countably compact and $f\colon X \to Y$ is continuous. Let $\mathcal{O}$ be a countable open cover of $f(X)$. Then $\{f^{-1}(U) \mid U \in \mathcal{O}\}$ is a countable open cover of X, hence admits a finite subcover $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$. Then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$.
This satisfies the extreme value theorem by the same reasoning as for sequential compactness; namely, that $\mathbb{R}$ is metrizable, so a countably compact subset of $\mathbb{R}$ is compact.
Limit point compactness: This is **not** preserved by continuous functions: let $X = \mathbb{Z} \times \{0,1\}$ where $\mathbb{Z}$ has the discrete topology and $\{0,1\}$ has the trivial topology. Then X is limit point compact, as *every* non-empty $S \subseteq X$ has a limit point. (Choose any point $(n,i) \in S$; then $(n, 1-i)$ is a limit point of S, as every neighbourhood of $(n, 1-i)$ contains the set $\{n\} \times \{0,1\}$, which in particular includes (n,i).) However, the image of X under the projection $f\colon X \to \mathbb{Z}$ is all of $\mathbb{Z}$, which is not limit point compact.
This example also shows that limit point compactness does **not** satisfy the extreme value theorem (as the discrete topology on $\mathbb{Z}$ agrees with the subspace topology inherited from the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$).

Problem 2(b)

Show that sequential compactness is countably productive but not arbitrary productive.
Hint: Show that $[0,1]^{\mathbb{R}}$ is not sequentially compact.

Solution

Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable collection of sequentially compact spaces and set $X = \prod_{i=1}^{\infty} X_i$. Fix a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X; we want to show that this has a convergent subsequence. First, a notational convention: we denote the entire sequence with the boldface symbol $\mathbf{x}_0 = \{x_n\}_{n\in\mathbb{N}}$, and will do similarly for other sequences in consideration.
First, choose a subsequence $\mathbf{x}_1 = \{x_{1,n}\}_{n\in\mathbb{N}}$ of $\mathbf{x}_0$ whose first coordinates form a convergent sequence in X_1, i.e., such that $\{\pi_1(x_{1,n})\}_{n\in\mathbb{N}}$ converges to some point in X_1 as $n \to \infty$. Next, choose a subsequence $\mathbf{x}_2 = \{x_{2,n}\}_{n\in\mathbb{N}}$ of $\mathbf{x}_1$ whose second coordinates converge in X_2. Continue in this way to obtain, for each $i \in \mathbb{N}$, a subsequence $\mathbf{x}_i = \{x_{i,n}\}_{n\in\mathbb{N}}$ of the previous sequence $\mathbf{x}_{i-1}$ whose i^{th} coordinate converges in X_i. Now consider the sequence obtained from the "diagonal" in the following array:

$$\begin{array}{rccccc}
\mathbf{x}_1 = & \color{blue}{x_{1,1}} & x_{1,2} & x_{1,3} & \cdots \\
\mathbf{x}_2 = & x_{2,1} & \color{blue}{x_{2,2}} & x_{2,3} & \cdots \\
\mathbf{x}_3 = & x_{3,1} & x_{3,2} & \color{blue}{x_{3,3}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \color{blue}{\ddots}
\end{array}$$

For each $i \in \mathbb{N}$, the tail of the sequence $\{x_{n,n}\}_{n\in\mathbb{N}}$ is eventually a subsequence of $\mathbf{x}_i$, and therefore converges in the i^{th} coordinate. Thus this diagonal sequence converges in each

coordinate, yielding the desired convergent subsequence of the original sequence.
To show that $[0,1]^{\mathbb{R}}$ is not sequentially compact, consider the sequence of functions $f_n\colon \mathbb{R} \to [0,1]$ defined by $f_n(x) =$ the n^{th} digit after the decimal point in the binary expansion of x. Given any subsequence f_{n_k}, let $x \in [0,1]$ be defined by the property that $f_{n_k}(x) = 0$ if k is odd and $f_{n_k}(x) = 1$ if k is even. Then the sequence $\{f_{n_k}(x)\}$ alternates between 0 and 1, and hence does not converge.

Problem 2(c)

Let $\mathcal{B}$ be the collection of all sets of the form $\{2n-1, 2n\}$ for $n \in \mathbb{N}$. Show that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{N}$. Decide which notion of compactness $(\mathbb{N}, \mathcal{T})$ satisfies.

Solution

$\mathcal{B}$ is a basis: We certainly have $\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{2n-1, 2n\}$. If $B_1, B_2 \in \mathcal{B}$ have non-empty intersection, then necessarily $B_1 = B_2$ as the sets in $\mathcal{B}$ are all mutually disjoint.
Claim: $(\mathbb{N}, \mathcal{T})$ is limit point compact, but not compact, sequentially compact, or countably compact.

Proof of claim. Observe that $\mathcal{B}$ itself is a countable open cover of $(\mathbb{N}, \mathcal{T})$ which does not admit a finite subcover. Thus $(\mathbb{N}, \mathcal{T})$ is not countably compact, which further implies that $(\mathbb{N}, \mathcal{T})$ is neither sequentially compact nor compact.
To show that $(\mathbb{N}, \mathcal{T})$ is limit point compact, we will prove the stronger statement that every non-empty subset $S \subseteq \mathbb{N}$ has a limit point. Indeed, if $m \in S$, then either $m = 2n$ is even, in which case $m-1$ is a limit point of S, or $m = 2n-1$ is odd, in which case $m+1$ is a limit point of S. (*Exercise:* Show that $(\mathbb{N}, \mathcal{T})$ is homeomorphic to the space $\mathbb{Z}_{\text{discrete}} \times \{0,1\}_{\text{trivial}}$ considered in part (a).) ■

Problem 2(d) (bonus)

Find an example of a sequentially compact space that is not compact.

Solution

Many choices are possible. For instance, the first uncountable ordinal, and the subspace

$$\{f \in \{0,1\}^{\mathbb{R}} \mid f^{-1}(1) \text{ is countable}\}$$

of $\{0,1\}^{\mathbb{R}}$ (see this StackExchange post).

Question 3

Problem 3(a)

Let $X := \{x_n \mid n \in \mathbb{N}\}$ be a countable metric space. Suppose X has no isolated points. Show that X cannot be sequentially compact by constructing a sequence in X that has no converging subsequence. Conclude that a compact metric space with no isolated point must be uncountable.

Solution

Let U_1 and V_1 be disjoint neighbourhoods of x_1 and x_2, respectively. Let J_1 be all the natural numbers $n > 2$ in which $x_n \in V_1$. Since X is T_1 and every point in X is a limit point, every neighbourhood of every point is infinite; in particular, J_1 is infinite.

Let $n_1 := 2$ and $n_2 := \min J_1$. Note that J_1 is a subset of $\mathbb{N}$ and so has a least element by the well ordering principle. Then let U_2 and V_2 be disjoint neighbourhoods of x_{n_1} and x_{n_2} contained in V_1. Proceed inductively as follows. Suppose x_{n_k}, U_k, and V_k are defined for some $k \in \mathbb{N}$. Then let $n_{k+1} = \min J_k$, where $J_k := \{n \in \mathbb{N} \mid n > n_k,\ x_n \in V_k\}$. Choose disjoint neighbourhoods U_{k+1} and V_{k+1} of x_{n_k} and $x_{n_{k+1}}$ contained in V_k. We have then obtained a sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ and two collections of open sets $\{U_k\}_{k\in\mathbb{N}}$ and $\{V_k\}_{k\in\mathbb{N}}$ such that $x_{n_k} \in V_k$, $U_k \cap V_k = \emptyset$ and $V_{k+1} \subseteq V_k$ for every $k \in \mathbb{N}$. Furthermore, it holds that $n_{k+1} := \min J_k$, where $J_k := \{n \in \mathbb{N} \mid n > n_k,\ x_n \in V_k\}$.

We claim that the sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ does not have a converging subsequence. Suppose there exists a subsequence $\{x_{n_{k_l}}\}_{l\in\mathbb{N}}$ that converges to $x_m \in X$ for some $m \in \mathbb{N}$. This implies that $x_m \in V_k$ for every $k \in \mathbb{N}$ since $\{V_k\}_{k\in\mathbb{N}}$ is nested. Choose $l \in \mathbb{N}$ such that $m < n_{k_l}$. Then $x_m \notin V_{k_{l+1}}$, which is a contradiction. We conclude that X is not sequentially compact.

Observe that if $X = \mathbb{Q} \cup (0, 1)$, then $\{x_{n_k}\}_{k\in\mathbb{N}}$ could be a sequence converging to π and so does not have a converging subsequence in X.

Since compactness is equivalent to sequential compactness in metric spaces, we have then proven that every countable metric space with no isolated points is not compact. Equivalently, every compact metric space with no isolated points is uncountable.

Problem 3(b)

Show that the Cantor set $\mathcal{C}$ is an uncountable, compact, totally disconnected space with no isolated points, but with empty interior.

Solution

$\mathcal{C}$ is compact: Each set C_n in the construction of the Cantor set is a finite union of closed intervals of length 3^{-n}, and is thus closed; it follows that $\mathcal{C} = \bigcap_{n\in\mathbb{N}} C_n$ is an intersection of closed sets, hence closed. Since $\mathcal{C}$ is contained in $[0, 1]$, it is bounded, and thus $\mathcal{C}$ is compact by the Heine–Borel theorem.

$\mathcal{C}$ is totally disconnected: It suffices to show that $\mathcal{C}$ does not contain any open interval. If there existed an open interval $(a, b) \subseteq \mathcal{C}$ with $a < b$, then $(a, b) \subseteq C_n$ for all $n \in \mathbb{N}$. In particular, $(a,b) \subseteq C_n$ for $n \in \mathbb{N}$ satisfying $b-a < 3^{-n}$. But this is impossible, as C_n is a disjoint union of closed intervals of length 3^{-n}.

$\mathcal{C}$ has no isolated points: Given $x \in \mathcal{C}$ and $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $3^{-n} \le \varepsilon$. Since $x \in C_n$, x lies in one of the closed intervals $[x_1, x_2]$ of length 3^{-n} which form C_n. Thus $(x-\varepsilon, x+\varepsilon) \cap \mathcal{C}$ contains points other than x, so x is not an isolated point of $\mathcal{C}$.

$\mathcal{C}$ has empty interior: This follows from the fact that $\mathcal{C}$ contains no non-empty open intervals.

$\mathcal{C}$ is uncountable: By 3a, $\mathcal{C}$ is a compact metric space with no isolated points, and hence must be uncountable.

Problem 3(c)

Show that $\mathcal{C}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ is equipped with the discrete topology. Use this to show that $\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$.

Solution

Each $x \in [0,1]$ can be represented in its ternary expansion, which means that $x = \sum_{k=1}^{\infty} a_n 3^{-n}$ for some sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\{0,1,2\}$. It is easy to see that all the end points of the subintervals in each C_n, which are numbers of the form $m3^{-n}$, all have a unique ternary expansion that contains only 0s and 2s, and all other points in $[0,1]$ have a unique ternary expansion. It is also easy to see that C_n is precisely all numbers in $[0,1]$ with a ternary expansion containing no 1s in the first n terms. (Note that if x is not an end point of an subinterval in any C_n, then $a_n = \lfloor 3^n (x - \sum_{k=1}^{n-1} a_k 3^{-k}) \rfloor$). It follows that $x \in \mathcal{C}$ if and only if it has a ternary expansion containing no 1s.

We define $f\colon \mathcal{C} \to \{0,1\}^{\mathbb{N}}$: given $x \in \mathcal{C}$, write x in its unique infinite ternary representation consisting only of the digits 0 and 2; replace all instances of the digit 2 with the digit 1. Interpret the infinite sequence as an element of $\{0,1\}^{\mathbb{N}}$. For instance,

$$1 = 0.2\overline{2}_3 \mapsto (1,1,\ldots), \qquad \frac{2}{9} = 0.02\overline{0}_3 \mapsto (0,1,0,0,\ldots), \qquad \frac{1}{4} = 0.02\overline{02}_3 \mapsto (0,1,0,1,\ldots).$$

In a formula,

$$f\left(\sum_{n=1}^{\infty} a_n 3^{-n}\right) = \left(\frac{a_1}{2}, \frac{a_2}{2}, \ldots\right).$$

f is bijective: Its inverse $g\colon \{0,1\}^{\mathbb{N}} \to \mathcal{C}$ can be explicitly written down as

$$g(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} (2a_n) 3^{-n}.$$

f is continuous: We will use the universal property of products to prove this. Define $f_n\colon \mathcal{C} \to \{0,1\}$ by

$$f_n\left(\sum_{n=1}^{\infty} a_n 3^{-n}\right) = \frac{a_n}{2}.$$

Then f_n is continuous: the preimage of the open set $\{0\}$ is the set of numbers in $\mathcal{C}$ whose n^{th} digit in their infinite ternary expansion is 0, and the preimage of the open set $\{1\}$ is those whose n^{th} digit in their infinite ternary expansion is 2. These are both open sets in $\mathcal{C}$, as they can be written as the intersection of $\mathcal{C}$ with a union of 2^{n-1} open intervals of length $3^{-n} + \varepsilon$ for sufficiently small ε (depending on n). See Figure 3.1 for an example.

Since each $f_n \colon \mathcal{C} \to \{0, 1\}$ is continuous, it follows that there is a unique continuous map $\mathcal{C} \to \{0, 1\}^{\mathbb{N}}$ whose n^{th} coordinate function is f_n. This map is f itself, so f is continuous.

f is a homeomorphism: $\mathcal{C}$ is compact (by part (b)) and $\{0, 1\}^{\mathbb{N}}$ is Hausdorff (since it is a product of discrete spaces); a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism, whence the result.

$\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$: This is because $\mathcal{C}^{\mathbb{N}} \cong (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N} \times \mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \cong \mathcal{C}$.

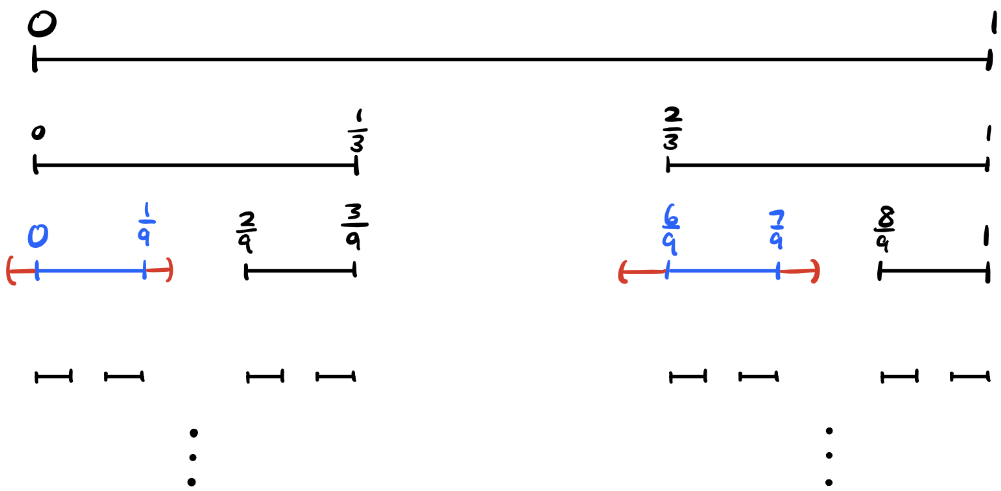

Figure 3.1: Numbers in $\mathcal{C}$ whose ternary expansion has a 0 in the second place must occur in the blue intervals. As such, $f_2^{-1}(\{0\})$ can be written as the intersection of $\mathcal{C}$ with the two open intervals in red, which shows that $f_2^{-1}(\{0\})$ is open.

Problem 3(d)

Show that $[0, 1]$, as well as $[0, 1]^{\mathbb{N}}$, is the continuous image of $\mathcal{C}$.

Solution

Define the map $g : \{0, 1\}^{\mathbb{N}} \to [0, 1]$ by $(a_n)_{n \in \mathbb{N}} \mapsto \sum_{n=1}^{\infty} a_n 2^{-n}$. g is surjective due to the fact that every number in $[0, 1]$ admits a binary expansion. It is continuous using an argument similar to the one described in 3(c) for f. Then $g \circ f$ is the desired continuous surjective function from $\mathcal{C}$ to $[0, 1]$. Taking the product map of this yields a continuous surjection $\mathcal{C}^{\mathbb{N}} \to [0, 1]^{\mathbb{N}}$, and pre-composing with the homeomorphism $\mathcal{C} \cong \mathcal{C}^{\mathbb{N}}$ from part (c) yields a continuous surjection $\mathcal{C} \to [0, 1]^{\mathbb{N}}$.

Problem 3(e)

Show that every compact metrizable space is the continuous image of $\mathcal{C}$.

Solution

X is second countable as it is compact and metrizable. Let $\{x_n \mid n \in \mathbb{N}\}$ be a countable dense set and let d be a metric inducting the topology on X and bounded by 1. Define the map $F : X \to [0,1]^{\mathbb{N}}$ by $F(x) = (d(x, x_n))_{n\in\mathbb{N}}$. We claim that F is an embedding. This then show that X is homeomorphic to a subset of $[0,1]^{\mathbb{N}}$. In light of 3d, this implies the existence of a continuous surjective function from $\mathcal{C}$ to X.

Since $x \mapsto d(x, x_n)$ is continuous for each $n \in \mathbb{N}$, which implies that F is continuous.
If $F(x) = F(y)$, then $d(x, x_n) = d(y, x_n)$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ large enough so that $d(x, x_n) < \epsilon/2$ and $d(y, x_n) < \epsilon/2$. Using the triangle inequality, $d(x, y) \leq d(x, x_n) + d(y, x_n) < \epsilon$. Since ϵ is arbitrary, it follows that $x = y$ and so F is injective. we conclude by the closed map lemma that F is a homeomorphism onto its image and, hence, an embedding as needed.

Question 4 (optional)

Let X be a non-compact, locally compact Hausdorff space. Recall that the topology on its one-point compactification $X^* = X \cup \{\infty\}$ is

$$\mathcal{T} = \{\text{open subsets of } X\} \cup \{U \subseteq X^* \mid X^* \setminus U \text{ is a compact subset of } X\}.$$

Note that this union is disjoint: sets in the first collection cannot contain the point at infinity, while sets in the second collection necessarily do contain ∞.

Problem 4(a)

Show that X^* is a compact Hausdorff space.

Solution

X^* **is compact:** Let $\mathcal{O} \subseteq \mathcal{T}$ be any open cover of X^*. For each $U \in \mathcal{O}$, let $U' = U \setminus \{\infty\}$ denote the same set as U, except possibly with the point at infinity removed (thus $U = U'$ if $U \subseteq X$). Then each U' is an open subset of X because either U was already an open subset of X, or $U = (X \setminus K) \cup \{\infty\}$ for some compact $K \subset X$, and hence $U' = X \setminus K$ is open in X (recall that compact sets are closed in Hausdorff spaces!). It follows that

$$\mathcal{O}' = \{U' \mid U \in \mathcal{O}\}$$

is an open cover of X.
Since $\mathcal{O}$ is an open cover of X^*, there exists $U_0 \in \mathcal{O}$ which contains the point ∞. Then $X^* \setminus U_0$ is a compact subset of X, so there exist finitely many $U_1, \ldots, U_n \in \mathcal{O}$ such that the corresponding $U'_1, \ldots, U'_n \in \mathcal{O}'$ cover $X^* \setminus U_0$. It follows that $\{U_0, U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{O}$.
X^* **is Hausdorff:** Fix distinct points $x \neq y$ in X^*.
Case 1. Suppose neither x nor y are the point at infinity. Then they are both points in X; since X is Hausdorff, there exist disjoint open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$. Since U and V are also open subsets of X^*, these yield the desired separating sets in X^*.
Case 2. Suppose one of them is the point at infinity, say $y = \infty$. Since X is locally compact, there exist subsets $U, K \subset X$ such that $x \in U \subseteq K$, U is open in X, and K is compact. Set

$V = (X \setminus K) \cup \{\infty\}$. Then U and V are the desired separating open subsets of X^*.

Problem 4(b)

Suppose that X is first countable. Show that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has no converging subsequence in X if and only if it converges to ∞ in X^*.

Solution

LEMMA 4.1. *If $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence in a locally compact Hausdorff space X, then there exists a compact subset of X which contains all points in the sequence, as well as the limiting point.*

Proof. Suppose $x_n \to x \in X$. Since X is locally compact, there exist $U, K \subseteq X$ such that $x \in U \subseteq K$, U is open in X, and K is compact. All but finitely many points in the sequence are contained in U, and taking the union of K with these finitely many points yields the desired compact set. ■

Unravelling the definitions, we have

$$\begin{gathered} x_n \to \infty \text{ in } X^* \\ \Updownarrow \\ \text{For every open neighbourhood } U \subseteq X^* \text{ of } \infty, \text{ the set } \{n \in \mathbb{N} \mid x_n \notin U\} \text{ is finite} \\ \Updownarrow \\ \text{For every compact } K \subset X, \text{ the set } \{n \in \mathbb{N} \mid x_n \notin (X \setminus K) \cup \{\infty\}\} \text{ is finite} \\ \Updownarrow \\ \text{For every compact } K \subset X, \text{ the set } \{n \in \mathbb{N} \mid x_n \in K\} \text{ is finite.} \end{gathered}$$

(Note that the last equivalence uses the fact that none of the x_n's are the point ∞.)
Suppose $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which converges to $x \in X$. By Lemma 4.1, there exists a compact $K \subset X$ containing all the x_{n_k}'s (and x). Then the set $\{n \in \mathbb{N} \mid x_n \in K\}$ is infinite, as it contains all the infinitely many n_k's. Thus $\{x_n\}_{n\in\mathbb{N}}$ does *not* converge to ∞ in X^*.
Conversely, suppose $\{x_n\}_{n\in\mathbb{N}}$ does not converge to ∞ in X^*. Then there exists a compact $K \subset X$ such that the set $\{n \in \mathbb{N} \mid x_n \in K\}$ is infinite. These indices define a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ which is contained inside K. Since X is first countable, so too is K (first countability is hereditary); thus K is also sequentially compact. Therefore by taking a further subsequence[a] which converges in K, we obtain a subsequence of the original sequence which converges in X.

[a] "subsubsequence"?

Problem 4(c)

Show that X is open in X^* and has the subspace topology. Show also that X is dense in X^*.

Solution

X is open in X^*: The topology on X^* includes all the open subsets of X, including X itself.

X has the subspace topology: Every open subset of X can be viewed as an open subset of X^* which does not contain the point at infinity, from which it follows that the original topology on X is coarser than the subspace topology induced by the inclusion $X \subset X^*$. On the other hand, as noted in the solution to part (a), removing the point at infinity from any open subset of X^* yields an open subset of X. Thus the subspace topology on X is coarser than the original topology on X. The two topologies therefore must coincide.

X is dense in X^*: This is equivalent to showing that every open neighbourhood $U \subseteq X^*$ of ∞ has non-empty intersection with X. This is true because such a neighbourhood must be of the form $U = (X \setminus K) \cup \{\infty\}$ for some compact $K \subset X$. Thus $U \cap X = X \setminus K$, which is non-empty as $K \neq X$ (recall that X is non-compact).

Problem 4(d)

Let N be the north pole in S^n. Show that the stereographic projection map $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ defined by $f(x_1, \ldots, x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n)$ is a homeomorphism that extends to a homeomorphism $\tilde{f}\colon S^n \to (\mathbb{R}^n)^*$.

Solution

We will first define $\tilde{f}\colon S^n \to (\mathbb{R}^n)^*$ by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \neq N, \\ \infty, & \text{if } x = N, \end{cases}$$

and show that $\tilde{f}$ is a homeomorphism. From this it automatically follows that $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is a homeomorphism, as the restriction of a homeomorphism is a homeomorphism (onto the restriction's image).

$\tilde{f}$ is bijective: This is equivalent to showing that $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$ is bijective, as $\tilde{f}$ maps N and no other point to ∞. For this, simply observe that the map $g\colon \mathbb{R}^n \to S^n \setminus \{N\}$ defined by

$$g(y_1, \ldots, y_n) = \left(\frac{2y_1}{\|y\|^2 + 1}, \ldots, \frac{2y_n}{\|y\|^2 + 1}, \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \right)$$

inverts f (here $\|\cdot\|$ denotes the Euclidean norm).

$\tilde{f}$ is continuous: The restriction of $\tilde{f}$ to $S^n \setminus \{N\}$ is given by f, which is visibly continuous from the formula (note that $x_{n+1} \neq 1$ for all $x \in S^n \setminus \{N\}$, so the denominator never vanishes). The only possible point of discontinuity is at N. To check continuity at this point, it suffices to check that if $\{X^k\}_{k \in \mathbb{N}}$ is a sequence in S^n converging to N, then $f(X^k) \to \infty$ in $(\mathbb{R}^n)^*$ as $k \to \infty$ (the spaces we are dealing with are first countable). This is true because $X^k \to N$ implies that $X^k_{n+1} \to 1$, and hence the $1 - X^k_{n+1}$ in the denominator converges to 0 as $k \to \infty$. Since $|X^k_i| \leq 1$ for all $i = 1, \ldots, n$, this implies that $\|f(X^k)\| \to \infty$, which is equivalent to $f(X^k) \to \infty$ in $(\mathbb{R}^n)^*$.

$\tilde{f}$ is a homeomorphism: A continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism. (Note that $(\mathbb{R}^n)^*$ is Hausdorff by part (a).)

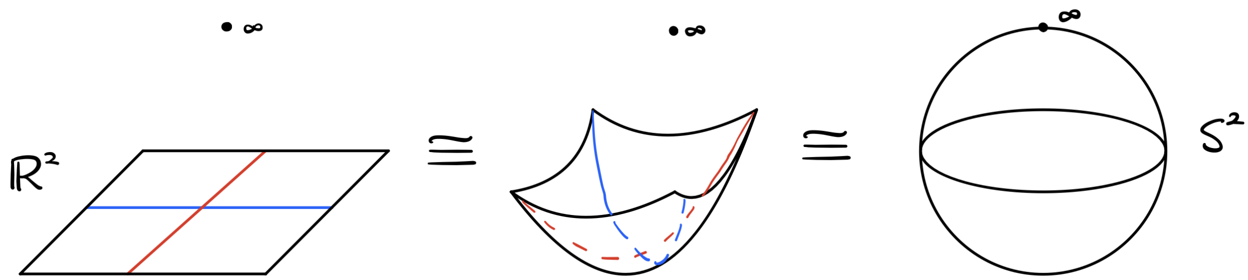

Figure 4.1: A cartoon depiction showing that $(\mathbb{R}^2)^* \cong S^2$.

Problem 4(e) (bonus)

Describe, geometrically if possible, the one-point compactifications of the following spaces: $\mathbb{R}^2 \setminus \{(0,0)\}$, $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$, $\bigcup_{n,m\in\mathbb{Z}} B_{1/2}((n,m))$ in $\mathbb{R}^2$, and $\mathbb{R}_{\text{discrete}}$.

Solution

Since $\mathbb{R}^2$ is homeomorphic to S^2 with a point removed, $\mathbb{R}^2 \setminus \{(0,0)\}$ is homeomorphic to S^2 with two points removed, say the north and south poles. The one-point compactification of $\mathbb{R}^2 \setminus \{(0,0)\}$ can thus be visualized by bringing the north and south poles of S^2 together to a single point; the result looks like a torus where the "inner circle" has been pinched to a point. This is known as a *horn torus*; see Figure 4.2.
Describing the one-point compactifications of the other three spaces is left as a healthy exercise to the reader.

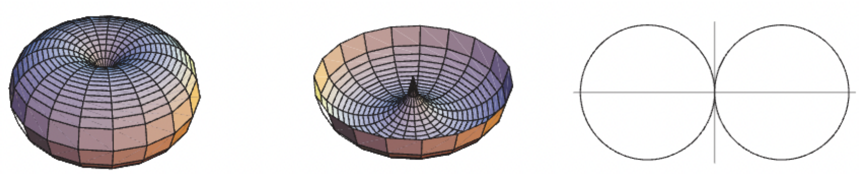

Figure 4.2: A full view (left), cutaway (middle), and cross-section (right) of a horn torus, which is homeomorphic to the one-point compactification of $\mathbb{R}^2 \setminus \{(0,0)\}$. Figure adapted from Wolfram MathWorld.