MAT 327: Introduction to Topology
Assignment #4
Due on Sunday July 16, 2023 by 11:59 pm

Note: This assignment covers material from Week #1-#7 and the first lecture from Week #8.

Problem 1

Let $f : (X, d_X) \to (Y, d_Y)$ be a function from a metric space to another. We say f is an isometry if $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$.
We say f is a shrinking map if $d_Y(f(x), f(y)) < d_X(x, y)$ for all $x, y \in X$ such that $x \neq y$.
We say f is a contraction if there is a number $\alpha \in [0, 1)$ such that $d_Y(f(x), f(y)) \leq \alpha d_X(x, y)$ for all $x, y \in X$.

(a) Let X be a compact metric space and $f : X \to X$ be an isometry. Show that f is a homeomorphism.

(b) Let X be a compact metric space and $f : X \to X$ be a shrinking map. Show that there exists a unique fixed point.

Remark: A fixed point is a point $x \in X$ such that $f(x) = x$.

(c) Let X be a complete metric space and $f : X \to X$ be a contraction. Show that there exists a unique fixed point.

(d) Show that if "compact" was replaced with "complete" in (b), the statement would be false.

Hint: Find a shrinking map $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point.

(e) ***(bonus)*** Would (a) be true if the codomain of f is a different compact metric space Y? What if we add the assumption that f is surjective?

Problem 2

We studied four formulation of compactness: compact, sequentially compact, countably compact, and limit point compact.

(a) Which of the four definitions of compactness is preserved by continuous functions? Which of the four definitions satisfy the extreme value theorem?

(b) Show that sequential compactness is countably productive but not arbitrary productive.

Remark: Tychonoff theorem, on the other hand, states that compactness is arbitrary productive. This gives us an example of a compact space that is not sequentially compact.
Hint: Show that $[0,1]^{\mathbb{R}}$ is not sequentially compact.

(c) Let $\mathcal{B}$ be the collection of all sets of the form $\{2n-1, 2n\}$ for $n \in \mathbb{N}$. Show that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{N}$. Decide which notion of compactness does $(\mathbb{N}, \mathcal{T})$ satisfy.

(d) ***(bonus)*** Find an example of a sequentially compact set that is not compact.

Remark: You might need to do some research first to answer this question.

Problem 3

In this problem, we will define the Cantor set, a classic and counterintuitive example in topology.

Let $C_0 = [0,1]$ with its usual topology. Let C_1 be the set obtained by deleting the open middle third of C_0, i.e. $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Let C_2 be the set obtained by deleting the open middle thirds of both of the intervals in C_1, i.e.

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Continue this process inductively, with C_n being the set obtained from C_{n-1} by deleting the open middle third of each of the intervals in C_{n-1}.
The Cantor set $\mathcal{C}$ is defined as $\mathcal{C} := \bigcap_{n \in \mathbb{N}} C_n$.

As a subset of $\mathbb{R}$, $\mathcal{C}$ is a second countable and metrizable.

(a) Let $X := \{x_n \mid n \in \mathbb{N}\}$ be a countable metric space. Suppose X has no isolated points. Show that X cannot be sequentially compact by constructing a sequence in X that has no converging subsequence. Conclude that a compact metric space with no isolated point must be uncountable.

Remark: The proof of the Baire Category theorem, which we will study soon, carries the same idea.
Remark: This particularly shows that $[0,1]$ is uncountable.

(b) Show that $\mathcal{C}$ is uncountable, compact totally disconnected space with no isolated points but with empty interior.

Remark: A space is totally disconnected if all the connected components are singletons.
Remark: Reflect on this for a while. This is a weird combination of properties satisfied by one space.

(c) Show that $\mathcal{C}$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$ where $\{0,1\}$ is equipped with the discrete topology. Use this to show that $\mathcal{C}$ is homeomorphic to $\mathcal{C}^{\mathbb{N}}$.

Hint: Convince yourself that $\mathcal{C}$ is precisely all the numbers in $[0,1]$ that have a ternary expansion (i.e. decimal expansion in base three) containing no 1s. These are all numbers in $[0,1]$ that are of the form $\sum_{n=1}^{\infty} a_n 3^{-n}$ where a_n is 0 or 2.
Remark: We can now think of $\mathcal{C}$ as $\{0,1\}^{\mathbb{N}}$ if we identify each number in $\mathcal{C}$ with its ternary expansion.

(d) Show that $[0,1]$, as well as $[0,1]^{\mathbb{N}}$, is the continuous image of $\mathcal{C}$.

Hint: Can you think of a map that sends the ternary expansion of a number in $\mathcal{C}$ to the binary expansion of another number in $[0,1]$?

(e) Show that every compact metrizable space is the continuous image of $\mathcal{C}$.

Hint: Think first of an embedding $F : X \to [0,1]^{\mathbb{N}}$ using the separability of X and a metric d that generates the topology on X that is bounded by 1. (An embedding is a map that is homeomorphic onto its image).

Problem 4 (optional)

Let X be a noncompact locally compact Hausdorff space. The one-point compactification of X is the topological space X^* defined as follows. Let ∞ be some point not in X, and let $X^* = X \cup \{\infty\}$ with the following topology:

$$\mathcal{T} = \{\text{open subsets of } X\} \bigcup \{U \subseteq X^* \mid X^* \setminus U \text{ is a compact subset of } X\}$$

Convince yourself that $\mathcal{T}$ is indeed a topology on X^*.

(a) Show that X^* is a compact Hausdorff space.

(b) Suppose that X is first countable. Show that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has no converging subsequence in X if and only if it converges to ∞ in X^*.

(c) Show that X is open in X^* and has the subspace topology. Show also that X is dense in X^*.

(d) Let N be the north pole in S^n and define the map $f : S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection defined by $f(x_1, ..., x_{n+1}) := \frac{1}{1-x_{n+1}}(x_1, ..., x_n)$. Show that f is a homeomorphism that extends to a homeomorphism $\tilde{f} : S^n \to (\mathbb{R}^n)^*$.

Remark: In complex analysis, the one-point compactification of $\mathbb{C}$ is often called the Riemann sphere; this problem shows that the Riemann sphere is homeomorphic to S^2.

(e) ***(bonus)*** Describe, geometrically if possible, the one-point compactifications of the following spaces:

- $\mathbb{R}^2 \setminus \{(0,0)\}$
- $\bigcup_{n,m\in\mathbb{Z}} B_{1/2}\big((n,m)\big)$ in $\mathbb{R}^2$.
- $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$
- $\mathbb{R}_{\text{discrete}}$