Assignment #3 Solutions

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1. a) *Proof.* Let V be an arbitrary subset of Y. I claim that the preimage $f^{-1}(V)$ is open in X.
For any $x \in f^{-1}(V)$, there exists an open neighborhood U on which f is constant. In particular, $f(U) = \{f(x)\} \subset V$, and $U \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open as desired.
Now, let x_0 be an element of X and $y_0 = f(x_0)$. By the claim above, $f^{-1}(y_0)$ is open. Furthermore, its complement $f^{-1}(y_0)^c = f^{-1}(Y - \{y_0\})$ is open. This shows that $f^{-1}(y_0)$ is clopen and non-empty subset of X. Since X is connected, we must have $f^{-1}(y_0) = X$. In other words, f is constant. □

b) *Proof.* Plugging in $x = 1$, we get in particular that $[x^2 f'(x)]'$ is positive at $x = 1$ and hence positive on $[1, 1+\epsilon)$ for some $\epsilon > 0$ by continuity of the function $x \mapsto x^2 f'(x)$. This implies that the function $x \mapsto x^2 f'(x)$ is increasing on $[1, 1+\epsilon)$ which implies that $f'(x) > 0$ on $(1, 1+\epsilon)$. Since $f(1) = 1$ and f is increasing on $(1, 1+\epsilon)$, it follows that $(1, 1+\epsilon) \subseteq A$ and so A is nonempty.

By virtue of the continuity of f' and f, A is open. Now suppose there exists a limit point of A that is not in A, then $b := \sup\{x > 1 \mid (1, x) \subseteq A\} < \infty$. Since $f'(x) > 0$ and $f(x) > 1$ for all $x < b$, it follows that $f'(b) \geq 0$ and $f'(b) \geq 1$. Since $f'(b) \geq 1$, it follows from the ODE that $[x^2 f'(x)]'$ is positive at $x = b$ and so the same holds on $(b - \delta, b + \delta)$ for some $\delta > 0$. Since $x^2 f'(x)$ is positive for $x < b$ and $x^2 f'(x)$ is increasing on $(b - \delta, b + \delta)$, it follows that $x^2 f'(x)$ as well as $f'(x)$ is positive on $(b - \delta, b + \delta)$. Since $f(x) > 1$ for $x < b$ and f is increasing on $(b - \delta, b + \delta)$, it follows that $f(x) > 1$ on $(b - \delta, b + \delta)$, which implies that $(b - \delta, b + \delta) \subseteq A$, contradicting the definition of b.

We conclude that A is a nonempty clopen subset of $(1, \infty)$ and hence is equal to $(1, \infty)$. In particular, f is increasing as desired.

□

c) *Proof.* Let S be a collection of disjoint open sets in a separable space X, and D be a countable dense subset of X. For each open set $U \in S$, the intersection $U \cap D$ is non-empty. Then for every open set U in S, we can pick a point x_U in $U \cap D$. We see that $U \mapsto x_U$ is injective as the sets in S are disjoint. This shows that $|S| \leq |D|$, and thus S is countable.

□

Let f be an increasing function, and a be a discontinuity. We have $f(x) < f(a)$ for all $x < a$. Since an increasing sequence bounded from above must be convergent, the left limit $\lim_{x \to a^-} f(x)$ exists. Similarly, the right limit $\lim_{x \to a^+} f(x)$ exists, and

$$\lim_{x \to a^-} f(x) \leq f(a) \leq \lim_{x \to a^+} f(x)$$

By assumption, a is a discontinuity. So,

$$\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$$

Observe that $I_a := (\lim_{x\to a^-} f(x), \lim_{x\to a^+} f(x))$ lies outside the image of f. Moreover, I_a and I_b are disjoint for distinct discontinuities a and b. It follows from the result above that f can have at most countably many discontinuities. The proof is similar for a decreasing function.

d) *Proof.* Let p be a cut-point of X and $h : X \to Y$ a homeomorphism. By definition, there exists disjoint open sets U and V open in the subspace $X \setminus \{p\}$ such that $X \setminus \{p\} = U \cup V$. We have

$$h(X \setminus \{p\}) = Y \setminus \{h(p)\} = h(U) \cup h(V)$$

The restriction of h to $X - \{p\}$ (as a function from $X - \{p\}$ to its image $Y - \{h(p)\}$) is also a homeomorphism. So, $h(U)$ and $h(V)$ are disjoint open sets that cover $Y \setminus \{h(p)\}$. Therefore $h(p)$ is a cut-point for Y and so h defines a bijection of cut-points. The number of cut-points is a topological invariant. □

Observe that all points in $\mathbb{R}$ are cut-points. However, if we remove 0 from $[0, \infty)$, we are left with $(0, \infty)$ which is still connected. All other points of $[0, \infty)$ are cut-points. And $\mathbb{R}^n$ has no cut points. It's impossible to establish a bijection of cut-points between any of the two spaces discussed above.

e) *Proof.* Recall that non-empty open sets of the co-finite (or co-countable) topology are sets whose complements are finite (resp. countable). If U and U^c are both non-empty and open in the co-finite topology, then U and U^c are both finite (resp. countable). But this is impossible because $\mathbb{R}$ is uncountable. □

Note that the proof above only relies on the space being uncountable. Since $\mathbb{R}$ minus a point is still uncountable, it is connected by the same proof. Hence, $\mathbb{R}_{co-finite}$ and $\mathbb{R}_{co-countable}$ do not have cut-points.
For the topologist's sine curve, points in $\{(x, \sin(1/x)) \,|\, 0 < x < 1)\}$ are cutpoints. On the other hand, the right endpoint $(1, \sin 1)$ is clearly not a cutpoint. Points in the interval $\{0\} \times [-1, 1]$ are also not cutpoints since any set B satisfying $A \subseteq B \subseteq \overline{A}$ is connected, where $A := \{(x, \sin(1/x)) \,|\, 0 < x < 1)\}$ and $\overline{A}$ is the topologist's sine curve.

2. a) *Proof.* Let D be the collection of all step functions with finitely many steps, rational step heights, and whose steps are all on rational intervals. It's clear that D is countable as the cartesian product of countable sets is countable.
Let U be an arbitrary open set, then $U = \prod_{x\in\mathbb{R}} U_x$, where $U_x \neq \mathbb{R}$ for finitely many x. We can find a function from D with step heights $f(x) \in U_x$ for these x's. Therefore D is dense in $\mathbb{R}^{\mathbb{R}}$.
Recall from $1(c)$ that a separable space has at most countably many disjoint open sets. In the box topology, however, we can construct an uncountable collection of disjoint open sets $\{U_x\}_{x\in\mathbb{R}}$ with

$$\pi_\alpha(U_x) = (\delta_{x\alpha} - 1, \delta_{x\alpha})$$

where $\delta_{x\alpha}$ is 1 when $x = \alpha$ and 0 otherwise. We can check that for $x \neq y$,

$$U_x \cap U_y = \prod_{\alpha\in\mathbb{R}} \pi_\alpha(U_x \cap U_y) = \prod_{\alpha\in\mathbb{R}} \pi_\alpha(U_x) \cap \pi_\alpha(U_y)$$

However, for $\alpha = x$,

$$\pi_x(U_x) \cap \pi_x(U_y) = (0, 1) \cap (-1, 0) = \varnothing$$

So, U_x and U_y are disjoint. □

b) *Proof.* Suppose a subset $B \subset A$ is countable. It suffices to show that B is not dense in A.

For a function $g \in A$, define $D_g := \{x \in \mathbb{R} \,|\, g(x) \neq f(x)\}$. By the definition of A, D_g is countable. Then,

$$D := \bigcup_{g \in B} D_g$$

A countable union of countable sets is, again, countable. But $\mathbb{R}$ is uncountable, so $D \neq \mathbb{R}$. Take $a \in \mathbb{R} - D$. It satisfies that $f(a) = g(a)$ for all $g \in B$.
Let U have $\pi_a(U) = \mathbb{R} - \{f(a)\}$, and $\pi_x(U) = \mathbb{R}$ for all $x \neq a$. Then U is an open set that doesn't intersect B. This shows that B is not dense. □

c) *Proof.* Suppose there exists a dense subset $A \subset \mathbb{R}^J$. Consider a set $\pi_\alpha^{-1}(\mathbb{R}_{>0})$ open in the product topology for $\alpha \in J$.
Define $f : J \to \mathcal{P}(A)$ to be

$$f(\alpha) = A \cap \pi_\alpha^{-1}(\mathbb{R}_{>0})$$

That is, $f(\alpha)$ contains all $u \in A$ that satisfies $u(\alpha) > 0$. Since A is dense, $f(\alpha)$ is non-empty.
Now, we verify that f is injective. That is, $f(\alpha) \neq f(\beta)$ whenever $\alpha \neq \beta$.
For $\alpha \neq \beta$, consider the nonempty open set $\pi_\alpha^{-1}(\mathbb{R}_{>0}) \cap \pi_\beta^{-1}(\mathbb{R}_{<0})$, which, by the density of A, contains $u \in A$ that satisfies $u(\alpha) > 0$ and $u(\beta) < 0$. So there exists $u \in A$ such that $u \in f(\alpha)$ but $u \notin f(\beta)$ implying that $f(\alpha) \neq f(\beta)$. This verifies that f is injective.
However, this implies $|J| \leq |\mathcal{P}(A)| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, which contradicts the assumption. □

3. a) Note that $P_2 \Rightarrow P_1$ and $P_4 \Rightarrow P_3$. We have

$$(P_2 \text{ but not } P_3) \Rightarrow (P_1 \text{ but not } P_3) \Rightarrow (P_1 \text{ but not } P_4)$$

and

$$(P_2 \text{ but not } P_3) \Rightarrow (P_2 \text{ but not } P_4)$$

So, it suffices to find a space that is P_2 but not P_3 (path-connected but not locally connected), and this space will satisfy all four cases of P_i but not P_j. We will see that the topologist's comb has this property.
It's clearly path-connected. However, it's not locally path-connected. Consider $x = (0, 1/2)$. Every element of the local basis of x contains infinitely many disconnected segments of "comb teeth".

Similarly, for the cases of P_j but not P_i, we have

$$(P_4 \text{ but not } P_1) \Rightarrow (P_3 \text{ but not } P_1) \Rightarrow (P_3 \text{ but not } P_2)$$

and

$$(P_4 \text{ but not } P_1) \Rightarrow (P_4 \text{ but not } P_2)$$

So, it suffices to find a space that is P_4 but not P_1 (locally path-connected but not connected).
For example, $(0, 1) \cup (2, 3)$. It's obviously not connected. To see that it is locally path-connected, take a small enough neighborhood so that it only intersects one of the two intervals.

b) *Proof.* Note that path-connectedness always implies connectedness. So, it suffices to show that a connected and locally path-connected space X is path-connected.
Let p be an arbitrary point of X. Define U_p to be the set of all points that can be connected to p by a path (so U_p is the path-connected component containing p). Since there is a path-connected local basis of p, U_p is certainly not empty.

We will show that U_p is also clopen.
To see that it is open, consider $q \in U_p$. By local path-connectedness, there exists an open neighborhood V of q that is path-connected. By connecting paths, points in V can also be connected to p by paths. Hence $V \subset U_p$.
To see that it is closed, consider a limit point l of U_p. By definition, there every neighborhood of l intersects U_p. We may choose the neighborhood to be path-connected. Again, by connecting paths, we see that $l \in U_p$.
Since X is connected and U_p is a non-empty clopen subset, we have $U_p = X$. □

4. a) *Proof.* By definition, the quotient topology is the finest topology that makes π continuous (i.e. $U \subset X/\sim$ is open implies $\pi^{-1}(U) \subset X$ is open).
Define
$$\mathcal{T} = \{U \subset X/\sim \mid \pi^{-1}(U) \text{ is open in } X\}$$
(i.e. U is open iff $\pi^{-1}(U)$ is open.)
If $\mathcal{T}$ is a topology, then it is the clearly the finest among all topologies that makes π continuous. So, it suffices to check that $\mathcal{T}$ is indeed a topology.
For $\varnothing$ and $X/\sim$,
$$\pi^{-1}(\varnothing) = \varnothing$$
$$\pi^{-1}(X/\sim) = X$$
Also, preimage of a finite intersection (resp. arbitrary union) is a finite intersection (resp. union) of preimages.
$$\pi^{-1}\left(\bigcap_{i=1}^{n} U_n\right) = \bigcap_{i=1}^{n} \pi^{-1}(U_j)$$
$$\pi^{-1}\left(\bigcup_{j\in J} U_n\right) = \bigcup_{j\in J} \pi^{-1}(U_j)$$
In conclusion, this shows that $\mathcal{T}$ is the quotient topology. By the definition of $\mathcal{T}$, it satisfies the statement in the question. □

b) *Proof.* Note that if $g = f \circ \pi$, then, more explicitly,
$$g(x) = (f \circ \pi)(x) = f([x])$$
Using $f([x]) = g(x)$ as the definition of f, it is indeed unique. This is well-defined as $g(x) = g(y)$ for all $y \in [x]$ (i.e. the definition doesn't depend on the choice of representative element x).
If f is continuous, then $g = f \circ \pi$ is a composition of continuous functions and is thus continuous.
Now suppose g is continuous. Let U be an open set. Then the preimage $g^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ is open. By part (a), this implies that $f^{-1}(U)$ is open. Hence, f is continuous. □

c) *Proof.* Define the equivalence relation by letting $0 \sim 1$. Every other point is only equivalent to itself. Define a function $g : [0,1] \to S^1$,
$$g(x) = (\cos(2\pi x), \sin(2\pi x))$$
We see that $g(0) = g(1)$. So, by part (b), this induces a continuous function $f : [0,1]/\sim \to S^1$.
Since g is bijective from $(0,1)$ to $S^1 - \{(1,0)\}$ and $g(0) = g(1) = (1,0)$, we can infer that f is bijective.
$[0,1]$ is compact and π is continuous, so $[0,1]/\sim = \pi([0,1])$ is compact. $\mathbb{R}^2$ is Hausdorff, so S^1 as a subspace is also Hausdorff. Since f is a bijective continuous function from a compact space to a Hausdorff space, it is a homeomorphism. □

g) First observe that $X/\sim$ is homeomorphic to a disk D^2 with an equivalence relation that identifies every boundary point to its antipodal point (let' call it $\sim_1$. $x \sim_1 -x$ for $x \in \partial D^2$). And this is homeomorphic to a closed upper hemisphere (i.e. $S^2_{\geq 0} := \{(x_1, x_2, x_3) \in S^2 \,|\, x_3 \geq 0\}$) with the same equivalence relation on the equator. See figure 1.

Figure 1

Also note that collapsing every line passing through the origin in $\mathbb{R}^3 \setminus \{0\}$ to one point is the same as first collapsing every ray to a point ($\sim_2$: $x \sim_2 y$ if $x = \lambda y$ for some $\lambda > 0$), and then identifying pairs of antipodal points ($\sim_3$: $x \sim_3 -x$). See figure 2.

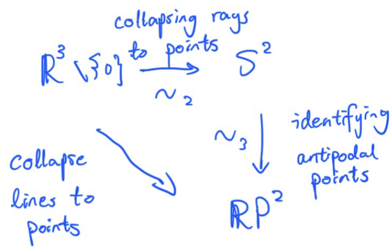

Figure 2

$(\mathbb{R}^3 \setminus \{0\})/\sim_2$ is just the 2-sphere S^2. So,

$$\mathbb{R}P^2 = ((\mathbb{R}^3 \setminus \{0\})/\sim_2)/\sim_3 = S^2/\sim_3$$

Let π_j denotes the projection corresponding to $\sim_j$. Let i denotes the inclusion map from $S^2_{\geq 0}$ to S^2. We see that $\pi_3 \circ i$ is a continuous function constant on equivalence classes of $\sim_1$ (i.e. pairs of antipodal points on the equator). By part (b), this induces a continuous function $f : S^2_{\geq 0}/\sim_1 \to \mathbb{R}P^2$ such that

$$\pi_3 \circ i = f \circ \pi_1$$

Note that f maps the equator of $S^2_{\geq 0}/\sim_1$ to the equator of $S^2/\sim_3$. For each x not on the equator, it maps x to the antipodal pair $\{x, -x\}$, and $-x$ is in the lower hemisphere. It's relatively easy to see that f is bijective.
Lastly, argue that f is a bijective continuous function from a compact space to a Hausdorff space, so it is a homeomorphism.

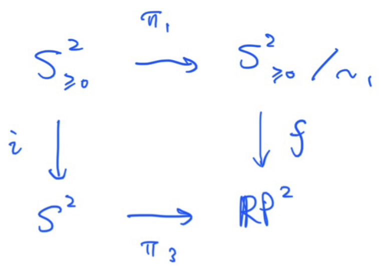

Figure 3