MAT 327: Introduction to Topology
Assignment #3
Due on Sunday June 18, 2023 by 11:59 pm

Note: This assignment covers material from Week #1-#5.

Problem 1

In this question, you will study some applications of connectedness and separability.

(a) Let X be a connected space and let $f : X \to Y$ be a locally constant function. Show that f is constant.

Remark: A function is locally constant if for every $x \in X$, there exists a neighbourhood U of x such that f is constant on U.

(b) Let $f : [1, \infty) \to \mathbb{R}$ be a smooth function satisfying

$$[x^2 f'(x)]' = \frac{1}{x^5}(f(x) - 1) + f(x)^3, \qquad f'(1) = 0, \quad f(1) = 1$$

Show that f is increasing.

Hint: Define the set $A := \{x \in (1, \infty) \mid f(x) > 1, f'(x) > 0\}$. Show that A is a nonempty clopen subset of $(1, \infty)$.

(c) Show that any separable space can have at most countably many disjoint open sets. Use this to show that any monotone function $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere except at most countably many points.

(d) For a connected space X, we say $p \in X$ is a cut-point if $X \setminus \{p\}$ is not connected. For a nonnegative integer n, show that having n cut-points is a topological invariant. Conclude that $[0, \infty)$ is not homeomorphic to $\mathbb{R}$, and $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ for $n > 1$.

Hint: Show that homeomorphisms send cut-points to cut-points.

(e) **(*bonus*)** Show that $\mathbb{R}_{co-finite}$ and $\mathbb{R}_{co-countable}$ are connected. Do they have cut points? What about the topologist's sine curve?

Problem 2

It is intuitive and easy to show that a countable product of separable spaces is separable. It is tempting to think that an uncountable product of separable spaces will not be separable

(a) Show that $\mathbb{R}^{\mathbb{R}}$ is separable. Can the same be said about $(\mathbb{R}^{\mathbb{R}}, \mathcal{T}_{box})$?

Hint: Consider the collection of all step functions with finitely many steps, rational step heights, and whose steps are all on rational intervals.

(b) This might help you partially regain your sanity. Show that there exists a subspace of $\mathbb{R}^{\mathbb{R}}$ that is not separable.

Hint: For $f \in \mathbb{R}^{\mathbb{R}}$, define the set $A := \{g \in \mathbb{R}^{\mathbb{R}} \mid g(x) = f(x)$ for all but countably many $x \in \mathbb{R}\}$

(c) ***(bonus)*** Show that $\mathbb{R}^J$ is not separable if $|J| > |\mathbb{R}|$.

Hint: Suppose $\mathbb{R}^J$ admits a countable dense set A. Find an injection from J to $P(A)$.

Problem 3

We will define the local analogue of connectedness and path-connected. We say a topological space X is locally connected (locally path-connected) if for every $x \in X$ and every neighbourhood U of x, there exists a connected (path-connected) open set V such that $x \in V \subseteq U$. Equivalently, X is locally connected (locally path-connected) if it admits a basis of connected (path-connected) sets.

We have then learnt four topological properties related to connectedness: connected (P_1), path-connected (P_2), locally connected (P_3), and locally path-connected (P_4). We have already shown that P_2 implies P_1, and so P_4 implies P_3.

(a) For each $(i, j) \in \{1, 2\} \times \{3, 4\}$, find a topological space satisfing P_i but not P_j and a topological space satisfying P_j but not P_i, or prove that no such space exists.

Hint: Here is a topological space that might be relevant. We define the topologist's comb as the subspace of $\mathbb{R}^2$ defined by $X := (I \times \{0\}) \cup (\{0\} \times I) \cup (A \times I)$ where $I = [0, 1]$ and $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

(b) Show that if X is locally path-connected, then connectedness is equivalent to path-connected. Conclude that every connected open set of $\mathbb{R}^2$ is path connected.

Problem 4

(only submit the parts marked with *)

Quotient spaces are spaces that are achieved by cutting, pasting and/or gluing other topological spaces to make another one. For example, a circle can be obtained from a closed interval by pasting the end points together.There are many ways one can formally define or construct quotient spaces. One way is to identify points in a topological space with each other (that is gluing the points together) to obtain a new smaller one.

Let X be a topological space and let $\sim$ be an equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes. Define the natural map $\pi : X \to X/\sim$ defined by $\pi(x) := [x]$. Equip $X/\sim$ with the finest topology that makes π continuous. We will call this the "quotient topology" on $X/\sim$ making it a "quotient space".

(a) * Show that $U \subset X/\sim$ is open iff $\pi^{-1}(U)$ is open.

Remark: A surjective map from a topological space to another topological space satisfying the above is called a quotient map.

(b) * Let Y be a topological space and $g : X \to Y$ be a map that is constant on equivalence classes. Show that there exists a unique map $f : X/\sim \to Y$ satisfying $g = f \circ \pi$. Show that f is continuous iff g is continuous.

(c) * Let $X = [0, 1]$. Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the circle S^1.

Hint: How do you get a circle from a closed interval? You glue the end points together.

Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$. We will glue different parts of X to construct different topological spaces.

(d) Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Hint: What happens when you glue together the left and right side of the square X?

Remark: When you glue together the left and right side of the square but in the opposite orientation, you get what we call the Möbius strip. Can you imagine what it looks like?

(e) Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the torus $S^1 \times S^1$.

Hint: After gluing the left and right side of X, consider gluing the top and bottom side (which are now circles).
Remark: If you glue the top and bottom circles in the opposite orientation, you get what we call the Klein bottle. Can you imagine what it looks like? No you can't; it's a 2 dimensional surface that cannot be visualized (i.e. embedded) in $\mathbb{R}^3$.

(f) We now let $X = \mathbb{R}^3 \setminus \{0\}$. We would like to define the space of all lines passing through the origin. So we wish to collapse every line to one point. Define an equivalence relation on X that accomplishes this. The quotient space you get is called the projective plane and is denoted by $\mathbb{RP}^2$.

(g) * ***(bonus)**** Consider the square X in $\mathbb{R}^2$ defined above. Glue the left and right side in the opposite orientation. Then glue the top and bottom sides in the opposite orientation. Show that the quotient space you get is $\mathbb{RP}^2$.