Assignment #2 Solutions

Jiawei Chen
email: jiaw.chen@mail.utoronto.ca

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1. a) The forward direction is obvious. Suppose the intersection of A and B is non-trivial. Then for $x \in A \cap B$, we have $0 \leq d(A, B) \leq d(x, x) = 0$ implying that $d(A, B) = 0$.

   For the other direction, consider the following lemmas:

   Lemma 1. *x is a limit point of A if and only if $d(x, A) = 0$.*

   Proof. By the sequence lemma, x is a limit point of A iff there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A that converges to x. In other words, we have

   $$\lim_{n\to\infty} d(x, a_n) = 0$$

   Furthermore, one can always find a sequence in A that converges to $d(x, A)$ by the properties of the infimum. Hence, if $d(x, A) = 0$, x is a limit point. Conversely, if x is a limit point, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in A converging to x; then

   $$0 \leq d(x, A) \leq \lim_{n\to\infty} d(x, a_n) = 0$$

   implying that $d(x, A) = 0$.

   □

   Lemma 2. *The notion of distance satisfies*

   $$d(A, B) \leq d(x, A) + d(x, B)$$

   for any $x \in X$.

   Proof. This follows from the triangle inequality of the metric. That is for any $x \in X$, $a \in A$ and $b \in B$,

   $$d(a, b) \leq d(x, a) + d(x, b)$$

   Take the infimum over all $a \in A$ and all $b \in B$ to obtain the result. □

   Corollary. *If x is a common limit point of A and B, then it follows from the lemmas above that*

   $$d(A, B) \leq d(x, A) + d(x, B) = 0$$

   So, $d(A, B) = 0$.

Now, return to the question. The other direction does not hold in general as disjoint A and B may still have common limit points.

Furthermore, this does not hold even under the assumption that A and B are closed. (The infimum of $d(A,B)$ doesn't need to be attained by a limit point.) Consider the following counterexample in $\mathbb{R}^2$:

$$A = \{(x,y)|y = 1/x\} \quad \text{and} \quad B = \{(x,y)|y = 0\}$$

. As x goes to infinity, $y = 1/x$ approaches 0.

b) Since a set is closed iff it contains all of its limit points, the statement directly follows from Lemma 1.

c) *Proof.* It is trivial to check $\bar{d}$ and d_0 are both bounded by 1. Now, we can check that (for any metric space) the ϵ-balls with radius $\epsilon < 1$ form a basis of the metric topology. Since the collection of balls with $\epsilon < 1$ is the same for $\bar{d}$ and d, they generate the same topology.

To see that $\bar{d}$ and d_0 are equivalent, observe that

$$\frac{1}{2}\bar{d} \leq d_0 \leq \bar{d}$$

□

2. a) *Proof.* "⇒":

Suppose x_n converges to x in the product topology. Let U_α be an open neighborhood of $\pi_\alpha(x)$. Then $\pi_\alpha^{-1}(U_\alpha)$ is an open neighborhood of x. By the definition of convergence, there exists N such $\pi_\alpha^{-1}(U_\alpha)$ contains all x_n for $n \geq N$.

This implies that there exists some N such that U contains all $\pi_\alpha(x_n)$ for $n \geq N$, which is exactly what we wanted to show.

"⇐":

Suppose $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x)$ for all $\alpha \in J$. Let U be an open neighborhood of x. Then $\pi_\alpha(U) = X_\alpha$ for all but finitely many indices. It's trivial that X_α contains all $\pi_\alpha(x_n)$. So, let's consider the indices for which the projection is not the whole space. Let these indices be $\beta_1, \beta_2, \ldots, \beta_m$, and the corresponding projections be $V_1, V_2, \ldots, V_m$.

For each V_j, there exists N_j such that $x_n \in V_j$ for all $n \geq N_j$. Taking $N = \max_{1\leq j\leq m} N_j$, we have $x_n \in U$ for $n \geq N$. Thus, x_n converges to x. □

Since the product topology is coarser than the box topology, convergence in the product topology is weaker. (As you can notice from the proof above: while the forward direction still works in the box topology, the backward direction fails as we don't have the restriction that $\pi_\alpha(U) = X_\alpha$ for all but finitely many indices. And N_α does not necessarily have an upper bound.)

b) This directly follows from part (a) by taking $X_\alpha = \mathbb{R}$ for all $\alpha \in J$ and $J = \mathbb{R}$. We interpret $x \in \mathbb{R}$ as an index. We understand the projection $\pi_x(f_n)$ as the function f_n evaluated at x. Then f_n converging pointwise to f means $\pi_x(f_n)$ converges to $\pi_x(f)$ for all x.

c) *Proof.* First, check that $\bar{d}$ satisfies the axioms of metric. (I will omit the details here.)

Note that the subspace topology is generated by sets of the form $B \cap C(\mathbb{R})$, with

$$B = \prod_{x\in\mathbb{R}} U_x$$

where $U_x = \mathbb{R}$ for all but finitely many x.

Let these x be $x_1, x_2, \dots, x_m$ and the corresponding projections $V_1, V_2, \dots, V_m$. Since each V_j is open, there exists an interval $(c_j - r_j, c_j + r_j) \subset V_j$. As there are only finitely many V_j, we can take $r = \min r_j$.

Let g be a continuous function with the interpolations $g(x_j) = c_j$. Then the r-ball (of the metric $\bar{d}$) centered at g lies inside of B. □

d) Convergence in $\bar{d}$ means that for all $\epsilon > 0$, there exists N s.t. $\bar{d}(f_n, f) < \epsilon$ for $n > N$. Recall that ϵ-balls with $\epsilon < 1$ is enough to form a basis. So, we can take $\epsilon < 1$ WLOG. In this case, $\bar{d}(f_n, f) = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$. Observe that the condition of convergence in $\bar{d}$ is exactly the same as the condition in the remark (also assuming $\epsilon < 1$).

e) Let f be a function in $\mathbb{R}^{\mathbb{R}}$, and B a set of the form described in part (c). Same as before, the projection of B is not the whole space for x_j with $1 \le j \le m$. (In particular, $f(x_j)$ lies in the projection.) Using Lagrange interpolation, we can construct a continuous function g such that $g(x_j) = f(x_j)$. Indeed, g lies in B. This shows that every open neighborhood of f intersects $C(\mathbb{R})$. $C(\mathbb{R})$ is dense in $\mathbb{R}^{\mathbb{R}}$.

However, this does not mean every function is a pointwise limit of continuous functions. The sequential definition of density only applies to metric spaces. In this case, $\mathbb{R}^{\mathbb{R}}$ is not metrizable. The function that is 1 at the rationals and 0 at the irrationals is an example of a function that is not the pointwise limit of continuous functions, but this is not easy to show. (Read about Baire class 1 and class 2 functions if you're interested).

3. a) *Proof.* Fix $m = k$, $\pi_k(x_n) = \frac{1}{kn}$, which converges to 0 as n goes to infinity. It follows from problem 2(a) that x_n convergence to the 0 sequence in the product topology.

For the box topology, consider

$$U = \prod_{n \in \mathbb{N}} (-x_{nn}, x_{nn})$$

where $x_{nn} := \pi_n(x_n)$. U is open in box topology and it contains the 0 sequence. However, for each sequence x_n, we have $x_{nn} \notin (-x_{nn}, x_{nn})$. This shows that U contains no member of $\{x_n\}_{n \in \mathbb{N}}$. The sequence does not converge to 0 in the box topology. □

b) (For convenience, I will use bold $\mathbf{x}$ to denote the sequence $\{x_n\}_{n \in \mathbb{N}}$)

Claim. $\mathbb{R}_0$ *is dense in* $\mathbb{R}^{\mathbb{N}}$ *with the product topology. (That is, the closure is* $\mathbb{R}^{\mathbb{N}}$*.)*

Proof. For any basis open set U, we can construct a sequence in $\mathbb{R}_0$ that lies in U. Basis open sets U is of the form

$$U = \prod_{n \in \mathbb{N}} U_n$$

where U_n is not the whole space for finitely many indices. Let $S \subset \mathbb{N}$ be the collection of such indices. We take x_n to lie inside U_n when $n \in S$. Otherwise, take $x_n = 0$. Then $\mathbf{x} \in U$ as desired. □

Claim. $\mathbb{R}_0$ *is closed in* $\mathbb{R}^{\mathbb{N}}$ *with the box topology.*

Proof. For any $\mathbf{x} \notin \mathbb{R}_0$, I will show there exists an open neighborhood U that does not intersect $\mathbb{R}_0$. By assumption, for any N, there exists $n > N$ such that $x_n \neq 0$. Let

$$U = \prod_{n \in \mathbb{N}} U_n$$

If $x_n \neq 0$, take U_n to be an open interval that contains x_n but not 0. If $x_n = 0$, take $(-1, 1)$. Since there exists $n > N$ such that U_n does not contain 0 for any N, this set does not intersect $\mathbb{R}_0$. □

Claim. *The closure of* $\mathbb{R}_0$ *is* c_0*, the collection of sequences that converge to 0, in* $\mathbb{R}^{\mathbb{N}}$ *with the uniform topology.*

Proof. First, I will show c_0 is closed. Suppose $\mathbf{x}$ does not converge to 0. Then there exists an ϵ that $\forall N\, \exists n > N$ s.t. $|x_n| > \epsilon$. Take a metric ball centered at $\mathbf{x}$ with radius $\epsilon/2$. Namely,

$$B_{\epsilon/2}(\mathbf{x}) = \{\mathbf{y} : \sup |x_i - y_i| < \epsilon/2\}$$

For any $\mathbf{y} \in B$, it follows that $\forall N\, \exists n > N$ s.t. $|y_n| > \epsilon/2$. So, $\mathbf{y}$ does not converge to 0. c_0 is indeed closed.
Now, I will show that $\mathbb{R}_0$ is dense in c_0. Let $\mathbf{z}$ be a member of c_0 and $B_\epsilon(\mathbf{z})$ with $0 < \epsilon < 1$ be a ball centered at $\mathbf{z}$. Since $\mathbf{z}$ converges to 0, there exists N s.t. $|z_n| < \epsilon$ for all $n > N$. Define a new sequence $\mathbf{z'}$ such that $z'_n = z_n$ when $n \leq N$, and $z' = 0$ otherwise. We see that $\mathbf{z'} \in B_\epsilon(\mathbf{z})$ and z' is eventually 0. Hence, $\mathbb{R}^{\mathbb{N}}$ is dense in c_0. This concludes the proof. □

c) *Proof.* First, notice that for any $i \in \mathbb{N}$,

$$|x_i - y_i|^2 \leq \sum_n |x_n - y_n|^2$$

Take the supremum of the left side and the inequality still holds. So, we have

$$\bar{\rho}(\mathbf{x}, \mathbf{y})^2 \leq \sup |x_n - y_n|^2 \leq \sum_n |x_n - y_n|^2 = d(\mathbf{x}, \mathbf{y})^2$$

Hence, $\bar{\rho}(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y})$ and $\mathcal{T}_{unif} \subset \mathcal{T}_{l^2}$.

Now, let $\mathbf{x}$ be a square-summable sequence and $B_\epsilon(\mathbf{x})$ a ball in l^2 metric.
Consider the open set of the box topology,

$$U = \prod_n (x_n - 2^{-n/2}\epsilon, x_n + 2^{-n/2}\epsilon)$$

For $y \in U$, we have

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2} < \sqrt{\epsilon^2 \sum_{n=1}^{\infty} 2^{-n}} = \epsilon$$

Hence, $U \subset B_\epsilon(\mathbf{x})$ and $\mathcal{T}_{l^2} \subset \mathcal{T}_{box}$ □

d) *Proof.* Since sequences in $\mathbb{R}_0$ have finitely many non-zero terms, they are indeed square-summable.

i) $\mathcal{T}_{l^2}$ and $\mathcal{T}_{box}$ are distinct on $\mathbb{R}_0$:
Consider

$$A = \mathbb{R}_0 \cap \prod_n (-1/n, 1/n)$$

which is open in the box topology. However, no matter how small ϵ is, we can always choose a small enough $k \in \mathbb{N}$ so that $1/k < \epsilon/2$, and

$$\mathbf{x} = (0, 0, ..., x_k = \epsilon/2, ..., 0, ...)$$

is inside the $B^d_\epsilon(\mathbf{0})$ but outside A. This shows that A is not open in the l^2 topology.

ii) $\mathcal{T}_{unif}$ and $\mathcal{T}_{l^2}$ are distinct on $\mathbb{R}_0$:
Consider

$$B_1^{l^2}(\mathbf{0}) = \left\{ \mathbf{x} \in \mathbb{R}_0 : \left(\sum_n |x_n|^2 \right)^{1/2} < 1 \right\}$$

No matter how small we choose ϵ to be. There is always some $k \in \mathbb{N}$ such that $k\epsilon^2/4 > 1$ and

$$\mathbf{x} = (\epsilon/2, \epsilon/2, ..., \epsilon/2, 0, 0, ...)$$

is inside $B_\epsilon^{\bar{\rho}}(\mathbf{0})$ but outside $B_1^{l^2}(\mathbf{0})$. This shows that $B_1^{l^2}(\mathbf{0})$ is not open in the uniform topology.

iii) $\mathcal{T}_{prod}$ and $\mathcal{T}_{unif}$ are distinct on $\mathbb{R}_0$:
Consider

$$B_1^{\bar{\rho}}(\mathbf{0}) = \{ \mathbf{x} \in \mathbb{R}_0 : \sup |x_n| < 1 \}$$

Let U be an open neighborhood of $\mathbf{0}$ in the product topology

$$U = \prod_n U_n$$

where U_n is the whole space for all but finitely many indices. Let k be a index such that U_k is the whole space, then

$$\mathbf{x} = (0, 0, ..., x_k = 2, ..., 0, ...)$$

is inside U but outside $B_1^{\bar{\rho}}(\mathbf{0})$. This shows that $B_1^{\bar{\rho}}(\mathbf{0})$ is not open in the product topology.

In conclusion, all four topologies are distinct on $\mathbb{R}_0$. □

e) See example 1 on Munkres p.132.