MAT 327: Introduction to Topology
Assignment #2
Due on Sunday June 4, 2023 by 11:59 pm

Note: This assignment covers material from Week #1-#3 and the Wednesday lecture from Week #4.

Problem 1

Let (X, d) be a metric space. A notion of distance between points in X gives us a notion of distance between sets as well as distance between points and sets. Let $A, B \subseteq X$ and let $x \in X$. We define

$$d(x, A) := \inf\{d(x, a) \mid a \in A\}$$
$$d(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}$$

Convince yourself that these notions are well defined by observing that we're taking the infimum of subsets of $\mathbb{R}$ that are bounded from below.

(a) Show that if $A \cap B \neq \phi$, then $d(A, B) = 0$, but the other direction doesn't hold. Will the other direction hold if we assume that A and B are closed?

(b) Show that A is closed iff $d(x, A) \neq 0$ for all $x \in A^c$.

(c) Define $\bar{d}, d_0 : X \times X \to [0, \infty)$ by $\bar{d}(x, y) := \min\{d(x, y), 1\}$ and $d_0(x, y) := \frac{d(x,y)}{d(x,y)+1}$ for $x, y \in X$. Show that $\bar{d}$ and d_0 are bounded metrics on X that generate the same topology as d.

Problem 2

(a) Let x_n be a sequence of points in the product space $\Pi_{\alpha \in J} X_\alpha$. Show that x_n converges to x iff $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x)$ in X_α for every $\alpha \in J$. Is convergence in the product topology stronger or weaker than convergence in the box topology?

(b) Convince yourself that the product space $\mathbb{R}^\mathbb{R}$ is the space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that a sequence of functions $f_n \in \mathbb{R}^\mathbb{R}$ converges to $f \in \mathbb{R}^\mathbb{R}$ iff f_n converges pointwise to f.

Remark: This means that $f_n(x)$ converges to $f(x)$ in the standard topology on $\mathbb{R}$ for every $x \in \mathbb{R}$.

(c) Define $C(\mathbb{R}) \subseteq \mathbb{R}^{\mathbb{R}}$ as the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define the metric $\bar{d} : C(\mathbb{R}) \times C(\mathbb{R}) \to [0, \infty)$ by

$$\bar{d}(f, g) := \min\{1, \sup_{x \in \mathbb{R}} |f(x) - g(x)|\}$$

Show that $\bar{d}$ is a metric that induces a topology on $C(\mathbb{R})$ that is finer than the subspace topology.

(d) Show that a sequence of functions $f_n \in C(\mathbb{R})$ converges to f in $(C(\mathbb{R}), \bar{d})$ iff f_n converges uniformly to f.

Remark: f_n converges uniformly to f if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n > N$ and $x \in \mathbb{R}$.

(e) ***(bonus)*** Show that $C(\mathbb{R})$ is dense in $\mathbb{R}^{\mathbb{R}}$. Does that mean that every function is the pointwise limit of continuous functions?

Problem 3

(a) Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$ by $x_n := \{\frac{1}{nm}\}_{m\in\mathbb{N}}$. Show that x_n converges to the 0 sequence in the product topology but not in the box topology.

Hint: Use problem 2a.

(b) Define $\mathbb{R}_0 \subset \mathbb{R}^{\mathbb{N}}$ to be the collection of sequences that are eventually constantly zero. Compute the closure of $\mathbb{R}_0$ in the product, uniform, and box topology.

A sequence $x = \{x_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is square-summable if $\sum_{n=1}^{\infty} x_n^2 < \infty$. The collection of all such sequences is denoted by ℓ_2 (pronounced little ell two to differentiate it from $L^2 \subseteq \mathbb{R}^{\mathbb{R}}$, which is the space of square integrable functions on $\mathbb{R}$). Define the metric

$$d(x, y) := \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$$

for sequences $x = \{x_n\}_{n\in\mathbb{N}}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ in ℓ_2 (convince yourself that d is indeed a metric on ℓ_2). This metric defines a topology on ℓ_2 called $\mathcal{T}_{\ell_2}$. We will denote by $\mathcal{T}_{box}$, $\mathcal{T}_{unif}$ and $\mathcal{T}_{prod}$ the subspace topologies that ℓ_2 inherits from the box, uniform, and product topologies on $\mathbb{R}^{\mathbb{N}}$, respectively.

(c) Show that $\mathcal{T}_{unif} \subseteq \mathcal{T}_{\ell_2} \subseteq \mathcal{T}_{box}$.

Remark: With what we proved in lectures, we then have that $\mathcal{T}_{prod} \subseteq \mathcal{T}_{unif} \subseteq \mathcal{T}_{\ell_2} \subseteq \mathcal{T}_{box}$.

(d) Show that $\mathbb{R}_0 \subseteq \ell_2$ and that the subspace topologies that $\mathbb{R}_0$ inherits from all four topologies are distinct.

(e) ***(bonus)*** Show that $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable.

Problem 4 (optional)

Quotient spaces are spaces that are achieved by cutting, pasting and/or gluing other topological spaces to make another one. For example, a circle can be obtained from a closed interval by pasting the end points together.There are many ways one can formally define or construct quotient spaces. One way is to identify points in a topological space with each other (that is gluing the points together) to obtain a new smaller one.

Let X be a topological space and let $\sim$ be an equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes. Define the natural map $\pi : X \to X/\sim$ defined by $\pi(x) := [x]$. Equip $X/\sim$ with the finest topology that makes π continuous. We will call this the "quotient topology" on $X/\sim$ making it a "quotient space".

(a) Show that $U \subset X/\sim$ is open iff $\pi^{-1}(U)$ is open and that π is an open map.

Remark: A surjective map from a topological space to another topological space satisfying the above is called a quotient map.

(b) Let Y be a topological space and $g : X \to Y$ be a map that is constant on equivalence classes. Show that there exists a unique map $f : X/\sim \to Y$ satisfying $g = f \circ \pi$. Show that f is continuous iff g is continuous.

(c) Let $X = [0, 1]$. Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the circle S^1.

Hint: How do you get a circle from a closed interval? You glue the end points together.

Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$. We will glue different parts of X to construct different topological spaces.

(d) Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Hint: What happens when you glue together the left and right side of the square X?
Remark: When you glue together the left and right side of the square but in the opposite orientation, you get what we call the Möbius strip. Can you imagine what it looks like?

(e) Define an equivalence relation on X such that the quotient space $X/\sim$ is homeomorphic to the torus $S^1 \times S^1$.

Hint: After gluing the left and right side of X, consider gluing the top and right side (which are now circles).
Remark: If you glue the top and right circles in the opposite orientation, you get what we call the Klein bottle. Can you imagine what it looks like? No you can't; it's a 2 dimensional surface that cannot be visualized (i.e. embedded) in $\mathbb{R}^3$.

(f) We now let $X = \mathbb{R}^3 \setminus \{0\}$. We would like to define the space of all lines passing through the origin. So we wish to collapse every line to one point. Define an equivalence relation on X that accomplishes this. The quotient space you get is called the projective plane and is denoted by $\mathbb{RP}^2$.

(g) ***(bonus)*** Consider the square X in $\mathbb{R}^2$ defined above. Glue the left and right side in the opposite orientation. Then glue the top and right sides in the opposite orientation. Show that the quotient space you get is $\mathbb{RP}^2$.