MAT 327 Assignment #1 Solutions

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1. a) *Proof.* "$\Rightarrow$":
Let B be a basis element in $\mathcal{B}$. For every $x \in B$, there exists $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subset B$. We can write B as a union of sets in $\mathcal{B}'$,

$$B = \bigcup_{x \in B} B'_x$$

"$\Leftarrow$":
Let B be a basis element in $\mathcal{B}$, and $\{B'_j\}_{j \in J}$, indexed by J, be a collection of sets in $\mathcal{B}'$ such that

$$B = \bigcup_{j \in J} B'_j$$

Indeed, for any $x \in B$, there exists some $j \in J$ such that $x \in B'_j \subset B$. □

b) *Proof.* Let $\|\cdot\|$, $\|\cdot\|'$ be two norms on $\mathbb{R}^n$. It is provided that they satisfy

$$\frac{1}{C}\|x\|' \leq \|x\| \leq C\|x\|'$$

for some constant $C > 0$.
It follows that for any radius $r > 0$ and center x, we have the following relation for the open balls

$$B_{r/C}(x) \subset B'_r(x) \tag{1}$$

Indeed, for any $y \in B_{r/C}(x)$, we have

$$\|x - y\|' \leq C\|x - y\| < r$$

That is, y is in $B_r(x)$ as well.
Moreover, for any $z \in B'_r(x)$, we can choose radius $l \leq r - \|x - z\|'$ so that

$$B'_l(z) \subset B'_r(x) \tag{2}$$

Combining (1) and (2), we have

$$B_{l/C}(z) \subset B'_l(z) \subset B'_r(x)$$

Thus, it follows from lemma 1.33 that the topology of $\|\cdot\|$ is finer than that of $\|\cdot\|'$.
To obtain the reverse containment, observe that,

$$\frac{1}{C}\|x\|' \leq \|x\| \leq C\|x\|'$$

implies

$$\frac{1}{C}\|x\| \leq \|x\|' \leq C\|x\|$$

□

c) For example, the following metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \tag{3}$$

generates the discrete topology on $\mathbb{R}^n$.

2. a) *Proof.* Verify the axioms for $\mathcal{T} = \{(a, \infty | a \in \mathbb{R})\} \cup \{\varnothing, \mathbb{R}\}\}$:

   1) Contains $\varnothing$ and $\mathbb{R}$: trivial
   2) Closure under unions:
      Let $\{a_j\}_{j \in J}$ be a collection of real numbers, then

$$\bigcup_{j \in J} (a_j, \infty) = (\inf_{j \in J} a_j, \infty) \in \mathcal{T}$$

   3) Closure under finite intersections:
      For a finite collection of reals $a_1 \leq a_2 \leq ..., \leq a_n$,

$$\bigcap_{i=1}^{n} (a_i, \infty) = (a_n, \infty) \in \mathcal{T}$$

□

b) *Proof.* It's relatively easy to see that the collection of finite intersections in $\mathcal{C}$ is $\mathcal{B} := \mathcal{C} \cup \mathcal{B}_{std} \cup \{\varnothing\}$, where $\mathcal{B}_{std} := \{(a,b) \mid a < b\}$ is the standard basis for $\mathbb{R}$. To see that $\mathcal{B}$ forms a basis for a topology, we show that it satisfies the two axioms in the definition of a basis:

   1. The union of all sets in $\mathcal{B}$ is clearly $\mathbb{R}$.
   2. If $B_1, B_2 \in \mathcal{B}$, then they are finite intersections of sets in $\mathcal{C}$. In particular, $B_1 \cap B_2$ is also a finite intersection of sets in $\mathcal{C}$ and so $B_1 \cup B_2 \in \mathcal{B}$.

Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$ and let $\mathcal{T}_{std}$ be the standard topology on $\mathbb{R}$. Since $\mathcal{B}_{std} \subseteq \mathcal{B}$, it follows that $\mathcal{T}_{std} \subseteq \mathcal{T}$.
Since sets of the form (a, ∞) and $(-\infty, a)$ for $a \in \mathbb{R}$ are clearly a union of open intervals and so are in $\mathcal{T}_{std}$, it follows that $\mathcal{B} \subseteq \mathcal{T}_{std}$. This implies that $\mathcal{T} \subseteq \mathcal{T}_{std}$ and hence $\mathcal{T} = \mathcal{T}_{std}$.

□

3. a) *Proof.* Observe that there is no $r > 0$ that satisfies $(a - r, a + r) \subset [a, b)$. So, $[a, b)$ is not open in the standard topology.
However, for any basis element of the standard topology (a, b), we can write

$$(a,b) = \bigcup_{c \in (a,b)} [c, b)$$

It follows from part (a) of problem 1 that the lower limit topology is finer than the standard topology. □

b) In the standard topology, the interior of A is $(0, 1)$ and the closure is $[0, 1]$.
In the lower limit topology, I claim that the interior and closure are, again, $(0, 1)$ and $[0, 1]$, respectively.
Interior: It follows from part (a) that $(0, 1)$ is open in $\mathbb{R}_l$. However, no basis element containing 1 lies completely inside A, so A is not open. Therefore the largest open set containing $(0, 1]$ is $(0, 1)$.
Closure: We can check that any basis element containing 0 would intersect A This shows that 0 is a limit point of A.
$A \cup \{0\} = [0, 1]$ is closed in the standard topology, so it is also closed in $\mathbb{R}_l$. We deduce that $[0, 1]$ is the smallest closed set containing A.

4. a) *Proof.* Recall that convergence of sequence means that for every open neighborhood U of x, there exists some N such that $x_n \in U$ for all $n > N$.
For the ray topology, open neighborhoods of x are of the form $(x - \epsilon, \infty)$ with $\epsilon > 0$. So, we can restate the above as

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} \text{ s.t. } x_n > x - \epsilon \text{ for all } n > N.$$

Recall (from tutorial 1) that this is exactly what $x \leq \liminf x_n$ means.

"f continuous $\Rightarrow \epsilon - \delta$ condition":
Since f is continuous, the preimage of $(f(x) - \epsilon, \infty)$ is open for every $x \in \mathbb{R}$. In particular, there exists some δ-ball $B_\delta(x)$ such that $f(B_\delta(x)) \subset (f(x) - \epsilon, \infty)$, which is precisely the $\epsilon - \delta$ condition provided in the question.

"f continuous $\Leftarrow \epsilon - \delta$ condition":
Let (a, ∞) be a basis element of the ray topology. Consider its preimage $f^{-1}((a, \infty))$. Suppose $x \in f^{-1}((a, \infty))$, then $f(x) > a$. Choose $\epsilon < f(x) - a$. Then, the $\epsilon - \delta$ condition gives an $\delta > 0$ such that $f(y) > f(x) - \epsilon > a$ for all $y \in B_\delta(x)$. So, $f(B_\delta(x)) \subset (a, \infty)$ (i.e. $B_\delta(x) \subset f^{-1}((a, \infty))$). In other words, there exists a δ-ball around x that lies inside $f^{-1}((a, \infty))$. This shows that $f^{-1}((a, \infty))$ is open, and hence f is continuous. □

b) $x_n \to x$ in the lower limit topology means that $x_n \to x$ in the usual sense and $x_n \geq x$ except for finitely many terms.

The only continuous functions from $\mathbb{R}_{std}$ to $\mathbb{R}_l$ are the constant functions. To see this, suppose the image of f contains two distinct values a and b (assume $a < b$). Take a number c such that $a < c < b$. Observe that $(-\infty, c)$ and $[c, \infty)$ are open (in $\mathbb{R}_l$), disjoint sets whose union is $\mathbb{R}$. Their preimages must also be open (in $\mathbb{R}_{std}$), disjoint sets whose union is $\mathbb{R}$. This can only be true if they are $\mathbb{R}$ and $\varnothing$. But this is a contradiction as $a \in (-\infty, c)$ and $b \in [c, \infty)$, the preimages should both be non-empty.

c)

	$\mathbb{R}_{std}$	$\mathbb{R}_l$	$\mathbb{R}_{ray}$
$a_n = 0$	$\{0\}$	$\{0\}$	$(-\infty, 0]$
$b_n = n$	$\{b_n\}$	$\{b_n\}$	$\mathbb{R}$
$c_n = -\frac{1}{n}$	$\{c_n\} \cup \{0\}$	$\{c_n\}$	$(-\infty, 0]$