MAT 327: Introduction to Topology
Assignment #1
Due on Sunday May 21, 2023 by 11:59 pm

Note: This assignment covers material from Week #1 and the Wednesday lecture from Week #2.

Problem 1

If we have two bases $\mathcal{B}$ and $\mathcal{B}'$ for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on a set X, then lemma 13.3 in Munkres's book allows us to compare the topologies $\mathcal{T}$ and $\mathcal{T}'$ by comparing their basis: $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for every basis element $B \in \mathcal{B}$ and for every $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

(a) Show that this is equivalent to the following:
Every set in $\mathcal{B}$ is a union of sets in $\mathcal{B}'$

Recall that a norm on $\mathbb{R}^n$ is a map $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ with the following properties: $\|x\| = 0$ implies $x = 0$, $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$. Given a norm on $\mathbb{R}^n$, we can define the metric $d(x, y) := \|x - y\|$. Convince yourself that this is indeed a metric on $\mathbb{R}^n$ as per the definition given in lectures.

A well known fact from analysis is that any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are equivalent in the sense that there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$,

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

(b) Use the above fact to show that the topology induced by a metric defined using a norm on $\mathbb{R}^n$ does not depend on the norm and is, in particular, the standard topology.

Hint: You can use **(a)** or lemma 13.3.

(c) Find a metric on $\mathbb{R}^n$ that induces a topology distinct from the standard topology.

Remark: This in particular shows that there are metrics on $\mathbb{R}^n$ that are not defined through a norm.

Problem 2

(a) Show that $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$ is a topology on $\mathbb{R}$.

Remark: This is called the ray topology on $\mathbb{R}$.

(b) Define the collection $\mathcal{C} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$. Show that the collection of finite intersection of sets in $\mathcal{C}$ forms a basis that generates the standard topology on $\mathbb{R}$.

Problem 3

You have shown in the post-lecture-practice-questions that the collection of sets of the form $[a, b)$ for $a, b \in \mathbb{R}$ forms a basis for a topology on $\mathbb{R}$ called the lower limit topology, denoted by $\mathbb{R}_\ell$.

(a) Show that $[a, b)$ is not open in the standard topology on $\mathbb{R}$. Conclude whether the standard topology is finer or coarser than the lower limit topology.

(b) Find the closure and interior of the set $A := (0, 1]$ in both the standard topology and the lower limit topology.

Problem 4

(a) Show that a sequence $x_n \in \mathbb{R}$ converges to $x \in \mathbb{R}$ in the ray topology iff $x \leq \liminf_{n\to\infty} x_n$. Furthermore, show that a function $f : (\mathbb{R}, \mathcal{T}_{std}) \to (\mathbb{R}, \mathcal{T}_{ray})$ is continuous iff for all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) < f(y) + \epsilon$ for any $y \in (x - \delta, x + \delta)$.

(b) ***(bonus)*** Describe what it means for a sequence $x_n \in \mathbb{R}$ to converge to $x \in \mathbb{R}$ in the lower limit topology. Furthermore, describe what it means for a function $f : (\mathbb{R}, \mathcal{T}_{std}) \to \mathbb{R}_\ell$ to be continuous.

(c) Define the following sequences: for $n \in \mathbb{N}$, $a_n := 0$, $b_n := n$, and $c_n := -\frac{1}{n}$. Find, without proof, the closures of $\{a_n \mid n \in \mathbb{N}\}$, $\{b_n \mid n \in \mathbb{N}\}$, and $\{c_n \mid n \in \mathbb{N}\}$ in the standard topology, the lower limit topology, and the ray topology.