MAT 327: Introduction to Topology
Instructor: Ahmed Ellithy
Midterm
Friday, June 23, 2023

Note: Submit by 11:30 AM through Crowdmark. No late submissions will be accepted. *You can get up to 58/50 in this test.*

Problem 1 [5]

(a) Define $\mathcal{T} := \{A \subseteq \mathbb{R} \mid 5 \in A\} \cup \{\emptyset\}$. Show that this is a topology on $\mathbb{R}$, denoted by $\mathbb{R}_5$, making it a separable connected topological space.

(b) Find a sequence in $\mathbb{R}_5$ that converges to every point. Conclude that $\mathbb{R}_5$ is not Hausdorff. Is it metrizable?

(c) ***(bonus)*** [**2**] Show that for every topological space X, there exists another topological space X' that is separable and contains X as a subspace.

Hint: Consider adding a new point to X and use $\mathbb{R}_5$ as an inspiration.

Problem 2 [5]

Let J be an uncountable set. Define $A \subseteq \mathbb{R}^J$ as follows

$$A := \{(x_\alpha)_{\alpha \in J} \in \mathbb{R}^J \mid x_\alpha = 0 \text{ for all but finitely many } \alpha \in J\}$$

Show that A is path connected and is dense in $\mathbb{R}^J$. Conclude that $\mathbb{R}^J$ is connected.

Problem 3 [10]

(a) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty closed sets in a topological space X that are nested in the sense that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. If one of the A_n's is compact, show that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

(b) Let X be a compact locally path-connected space. Show that X can be partitioned into a disjoint union of finitely many connected open sets.

Problem 4 [5]

(a) Define the function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ by

$$f(t) := (t, t^2, t^3, ...)$$

Determine whether f is continuous when $\mathbb{R}^{\mathbb{N}}$ is equipped with the product, uniform and box topologies.

(b) ***(bonus)*[2]** It is not true that a function $f : \mathbb{R} \to (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{unif})$ is continuous if and only if the components $\pi_n \circ f : \mathbb{R} \to \mathbb{R}$ are continuous. Find an example of a function f that illustrates this.

Problem 5 [5]

Let $\sim$ be an equivalence relation on a second countable topological space X such that the natural map $\pi : X \to X/\sim$ is an open map. Suppose that $\Delta := \{(x, y) \in X \times X \mid x \sim y\}$ is closed in $X \times X$. Show that the quotient space $X/\sim$ is Hausdorff and second countable.

Problem 6 [20]

Are the following true or false? Justify your answer briefly.
There are 10 questions, 3 marks each; 20 is the maximum mark (excluding the bonus).

- **(a)** The map $f : [0,1) \to S^1$ defined by $f(t) := (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism.
- **(b)** Let $A_\alpha \subseteq X_\alpha$ for every $\alpha \in J$. Then $\mathrm{Int}(\Pi_{\alpha\in J} A_\alpha) = \Pi_{\alpha \in J}(\mathrm{Int} A_\alpha)$.
- **(c)** Let $X = \mathbb{R} \setminus \{0\}$. Then $A \subseteq X$ is compact if and only if it's closed and bounded (with respect to the euclidean metric).
- **(d)** X is connected if and only if the only subsets with empty boundary are $\emptyset$ and X.
- **(e)** Let $(X, \mathcal{T})$ be a topological space. Let d be a metric on X that is continuous as a map from $X \times X$ to $\mathbb{R}$ when X is equipped with $\mathcal{T}$. Then the metric topology with respect to d is finer than $\mathcal{T}$.
- **(f)** Let X be a first countable with a countable dense set A. For each $a \in A$, let $\mathcal{N}_a$ be a countable neighbourhood basis at a. Then $\bigcup_{a\in A} \mathcal{N}_a$ is a basis for X.
- **(g)** Let $f : S^1 \to \mathbb{R}$ be a continuous function. Then f is not injective.
- **(h)** Let X be a second countable space with the property that sequences converge to at most one point. Then X is Hausdorff.
- **(i)** $\{0,1\}^{\mathbb{N}}$ is metrizable where $\{0,1\}$ is equipped with the discrete topology.
- **(j)** Let $f : X \to Y$ be a surjective continuous function. If X is separable, then Y is too.
- **(k)** **(*bonus* [2])** A space X has k connected components if and only if there exists a surjective continuous function $f : X \to \{1, 2, ..., k\}$ where $\{1, 2, ..., k\}$ is equipped with the discrete topology.

Problem 7: *bonus* [2]

Let $f : X \to Y$ be a surjective continuous map where X is compact and Y is Hausdorff. Define an equivalence relation on X as follows: $x \sim y$ if $f(x) = f(y)$. Show that $X/\sim$ is homeomorphic to Y.

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