MAT 327: Introduction to Topology
Instructor: Ahmed Ellithy
Midterm
Friday, June 23, 2023

Note: Submit by 11:30 AM through Crowdmark. No late submissions will be accepted. *You can get up to 58/50 in this test.*

Problem 1 [5]

(a) Define $\mathcal{T} := \{A \subseteq \mathbb{R} \mid 5 \in A\} \cup \{\emptyset\}$. Show that this is a topology on $\mathbb{R}$, denoted by $\mathbb{R}_5$, making it a separable connected topological space.

Clearly $\emptyset \in \mathcal{T}$ by definition and $\mathbb{R} \in \mathcal{T}$ since it contains 5. Furthermore, the union of sets containing 5 will also contain 5, and the finite intersection of sets containing 5 also contains 5. We conclude that $\mathcal{T}$ is a topology on $\mathbb{R}$. We will denote $(\mathbb{R}, \mathcal{T})$ by $\mathbb{R}_5$.

Let $U \subset \mathbb{R}_5$ be a clopen set. If $U \neq \emptyset$, then $5 \in U$. Since U is closed, U^c is open. Since $5 \notin U^c$, $U^c = \emptyset$ and so $U = \mathbb{R}_5$. We conclude that the only clopen sets are $\emptyset$ and $\mathbb{R}_5$ and hence $\mathbb{R}_5$ is connected.

Since every nonempty open set contains 5, $\{5\}$ is a countable dense set and so $\mathbb{R}_5$ is separable.

(b) Find a sequence in $\mathbb{R}_5$ that converges to every point. Conclude that $\mathbb{R}_5$ is not Hausdorff. Is it metrizable?

Consider the constant sequence $x_n = 5$. For $a \in \mathbb{R}_5$, every neighbourhood of a will contain 5 by definition of the topology, and so x_n converges to a. So x_n converges to every point. By a theorem in lectures, $\mathbb{R}_5$ is not Hausdorff since there exists a sequence converging to more than one point. Since every metric space is Hausdroff, $\mathbb{R}_5$ is not metrizable.

(c) ***(bonus)*** [**2**] Show that for every topological space X, there exists another topological space X' that is separable and contains X as a subspace.

Hint: Consider adding a new point to X and use $\mathbb{R}_5$ as an inspiration.

Define $X' := X \cup \{p\}$ where p is a point not in X. Equip X' with the topology $\mathcal{T}' := \{U \cup \{p\} \mid U \in \mathcal{T}\} \cup \{\emptyset\}$ where $\mathcal{T}$ is the topology on X. Then X' is separable since $\{p\}$ is dense, and X is a subspace of X' since every open set U in X is the intersection of X with an open set in X', namely $U \cup \{p\}$.

Problem 2 [5]

Let J be an uncountable set. Define $A \subseteq \mathbb{R}^J$ as follows

$$A := \{(x_\alpha)_{\alpha \in J} \in \mathbb{R}^J \mid x_\alpha = 0 \text{ for all but finitely many } \alpha \in J\}$$

Show that A is path connected and is dense in $\mathbb{R}^J$. Conclude that $\mathbb{R}^J$ is connected.

Let $(x_\alpha)_{\alpha \in J} \in A$. Define the path $f : [0,1] \to A$ by $f(t) = (tx_\alpha)_{\alpha \in J}$. Since every component of f is continuous as a function from $[0,1]$ to $\mathbb{R}$, f is continuous. Then clearly f is a path from the zero point $\bar{0} \in A$ to $(x_\alpha)_{\alpha \in J}$. Since any point in A can be joined to $\bar{0}$ by a path in A, it follows that any two points in A can be joined by a path in A and so A is path-connected.

Let ΠU_α be a basis open set in $\mathbb{R}^J$. Let $\alpha_1, ..., \alpha_k$ be the points in J in which $U_\alpha = \mathbb{R}$ for $\alpha \neq \alpha_1, ..., \alpha_k$. Let $(x_\alpha)_{\alpha \in J}$ be the point in $\mathbb{R}^J$ such that $x_\alpha = 0$ for $\alpha \neq \alpha_1, ..., \alpha_k$ and $x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, ..., k$. It follows that $(x_\alpha)_{\alpha \in J} \in A \cap \Pi U_\alpha$, and so every basis open set in $\mathbb{R}^J$ intersects A. We conclude that A is dense. Since A is path connected, A is connected and hence $\bar{A} = \mathbb{R}^J$ is connected.

Remark: A similar argument to the one described in the first paragraph also shows that $\mathbb{R}^J$ is path-connected, which particularly implies that $\mathbb{R}^J$ is connected. But you are asked to show that $\mathbb{R}^J$ is connected by finding a dense path-connected set, which is a common way to show a space is connected.

Problem 3 [10]

(a) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty closed sets in a topological space X that are nested in the sense that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. If one of the A_n's is compact, show that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

Suppose A_k is compact for some $k \in \mathbb{N}$. Define $U_n := A_k \setminus A_n$, which is open in A_k. Suppose $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Then

$$A_k = A_k \setminus \big(\bigcap_{n\in\mathbb{N}} A_n\big) = \bigcup_{n\in\mathbb{N}} A_k \setminus A_n = \bigcup_{n\in N} U_n$$

Since $U_n = \emptyset$ for $n \leq k$, we have that $\bigcup_{n=k+1}^{\infty} U_n = A_k$. In particular, the collection $\{U_n\}_{n>k}$ is an open cover for the compact set A_k, and so there exists $k < n_1 < n_2 < ... < n_m$ such that $U_{n_1}, ..., U_{n_m}$ is a cover for A_k. This implies that $A_k = \bigcup_{i=1}^{i=m} U_{n_i}$. By taking A_k minus both sides and using the fact that $A_{n_m} \subseteq A_{n_i}$ for $i = 1, ..., m$, we have that $\emptyset = \bigcap_{i=1}^{i=m} A_{n_i} = A_{n_m}$. This is a contradiction since we know that A_n is nonempty for every n.

(b) Let X be a compact locally path-connected space. Show that X can be partitioned into a disjoint union of finitely many connected open sets.

Let $(C_\alpha)_{\alpha\in J}$ be the connected components for X. We claim that C_α is open for all $\alpha \in J$. For $\alpha \in J$ and $x \in C_\alpha$, we have that there exists a connected open neighbouhood B_x of x by virtue of X being locally connected. Since B_x is connected and C_α is the connected component containing x, it follows that $B_x \subseteq C_\alpha$ by the properties of connected components and hence C_α is open.

Since $(C_\alpha)_{\alpha\in J}$ is an open cover for X and X is compact, it follows that there exists finitely many $C_{\alpha_1}, ..., C_{\alpha_m}$ that cover X. Since connected components are disjoint and are connected, we have then shown that X can be partitioned into a disjoint union of finitely many connected open sets, namely $C_{\alpha_1}, ..., C_{\alpha_m}$.

Problem 4 [5]

(a) Define the function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ by

$$f(t) := (t, t^2, t^3, ...)$$

Determine whether f is continuous when $\mathbb{R}^{\mathbb{N}}$ is equipped with the product, uniform and box topologies.

The components of f are $\pi_n \circ f(t) = t^n$ and so they're continuous from $\mathbb{R}$ to $\mathbb{R}$. It follows that f is continuous with respect to the product topology

Let $U := f^{-1}(B_{\bar{\rho}}(\bar{1}, 1/2)$ where $\bar{1}$ is the constant sequence $x_n = 1$. Clearly $1 \in U$. Let $\epsilon > 0$. Then $|\pi_n \circ f(1+\epsilon) - 1| = (1+\epsilon)^n - 1$, which goes to ∞ as $n \to \infty$. This shows that $\bar{\rho}(f(1+\epsilon), \bar{1}) = 1$ and so $f(1+\epsilon) \notin B_{\bar{\rho}}(\bar{1}, 1/2)$ implying that $1+\epsilon \notin U$ for every $\epsilon > 0$. This shows that 1 is not an interior point of U and so U is not open. We conclude that f is not continuous with respect to the uniform topology.

Since the box topology is finer than the uniform topology and f is not continuous with respect to the uniform topology, we conclude that the same holds for the box topology. Another way to see this is by observing that $f^{-1}(\Pi_{n\in\mathbb{N}}(-\frac{1}{n^n}, \frac{1}{n^n})) = \{0\}$. This clearly holds since $t \in \mathbb{R}$ is contained in the left side if and only if $t^n \in (-\frac{1}{n^n}, \frac{1}{n^n})$ which implies that $t \in (-\frac{1}{n}, \frac{1}{n})$ and so $t = 0$.

(b) ***(bonus)*[2]** It is not true that a function $f : \mathbb{R} \to (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{unif})$ is continuous if and only if the components $\pi_n \circ f : \mathbb{R} \to \mathbb{R}$ are continuous. Find an example of a function f that illustrates this.

The function above is an example since the components are all continuous from $\mathbb{R}$ to $\mathbb{R}$ but f is not continuous with respect to the uniform topology.

Problem 5 [5]

Let $\sim$ be an equivalence relation on a second countable topological space X such that the natural map $\pi : X \to X/\sim$ is an open map. Suppose that $\Delta := \{(x, y) \in X \times X \mid x \sim y\}$ is closed in $X \times X$. Show that the quotient space $X/\sim$ is Hausdorff and second countable.

We first show that $X/\sim$ is second countable. Let $\mathcal{B}_1$ be a countable basis for X. We claim that $\mathcal{B}_2 := \{\pi(B) \mid B \in \mathcal{B}_1\}$ is a basis for $X/\sim$. Since π is an open map, all sets in $\mathcal{B}_2$ are open. Let U be an arbitrary open set in $X/\sim$ and $[x] \in U$. Then $\pi^{-1}(U)$ is an open set containing x and so there exists a basis set $B_x \in \mathcal{B}_1$ such that $x \in B_x \subseteq \pi^{-1}(U)$. Then $\pi(B_x) \subseteq \pi(\pi^{-1}(U)) = U$ since π is surjective, and so $\pi(B_x)$ is an element in $\mathcal{B}_2$ containing $[x]$ and is contained in U as needed.

We now show that $X/\sim$ is Hausdorff. Let $[x_0], [y_0] \in X/\sim$ such that $[x_0] \neq [y_0]$. In particular, this means that $(x_0, y_0) \notin \Delta$. Since Δ^c is open in $X \times X$, there exists a basis set $U \times V$ in $X \times X$ such that $(x_0, y_0) \in U \times V \subseteq \Delta^c$. This implies that x is not related to y for any $x \in U$ and $y \in V$. Equivalently, $[x] \neq [y]$ for all $x \in U$ and $y \in V$, and so $\pi(U)$ and $\pi(V)$ are disjoint. Since π is open, we conclude that $\pi(U)$ and $\pi(V)$ are open disjoint neighbourhoods of $[x_0]$ and $[y_0]$ respectively and so $X/\sim$ is Hausdorff.

Problem 6 [20]

Are the following true or false? Justify your answer briefly.
There are 10 questions, 3 marks each; 20 is the maximum mark (excluding the bonus).

(a) The map $f : [0,1) \to S^1$ defined by $f(t) := (\cos 2\pi t, \sin 2\pi t)$ is a homeomorphism.
False. S^1 is compact since it's a closed and bounded subset of $\mathbb{R}$ but $[0,1)$ is not compact and so they can't be homeomorphic.

(b) Let $A_\alpha \subseteq X_\alpha$ for every $\alpha \in J$. Then $\text{Int}(\Pi_{\alpha\in J} A_\alpha) = \Pi_{\alpha\in J}(\text{Int} A_\alpha)$.

False. For an infinite set J, $\Pi_{\alpha\in J}\text{Int}(0,1) = \Pi_{\alpha\in J}(0,1)$ is not open in $\mathbb{R}^J$ but $\text{Int}\big(\Pi_{\alpha\in J}(0,1)\big)$ is open; so they cannot be equal. (In fact, the later is the empty set.)

(c) Let $X = \mathbb{R} \setminus \{0\}$. Then $A \subseteq X$ is compact if and only if it's closed and bounded (with respect to the euclidean metric).

False. $[-1,0) \cup (0,1]$ is a closed bounded subset of X but is not compact.

(d) X is connected if and only if the only subsets with empty boundary are $\emptyset$ and X.

True. U is a clopen subset of X if and only if every point in X is either an interior point of U or an interior point of U^c if and only if U has no boundary. The statement follows as X is connected if and only if there are no proper clopen subsets of X.

(e) Let $(X, \mathcal{T})$ be a topological space. Let d be a metric on X that is continuous as a map from $X \times X$ to $\mathbb{R}$ when X is equipped with $\mathcal{T}$. Then the metric topology with respect to d is finer than $\mathcal{T}$.

False. Equip $X = \mathbb{R}$ with the discrete metric and let d be the euclidean metric. Then d is continuous but the standard topology is not finer than the discrete topology. In fact, the statement would be true if "finer" was replaced with "coarser".

(f) Let X be a first countable with a countable dense set A. For each $a \in A$, let $\mathcal{N}_a$ be a countable neighbourhood basis at a. Then $\bigcup_{a\in A} \mathcal{N}_a$ is a basis for X.

False. Let $X = \mathbb{R}_\ell$. Check the note on Piazza titled "Corrections to today's lecture (June 7)".

(g) Let $f : S^1 \to \mathbb{R}$ be a continuous function. Then f is not injective.

True. Since S^1 is compact and $\mathbb{R}$ is Hausdorff, then f is injective implies that it's a homeomorphism onto its image by the closed map lemma in lectures. Since S^1 is connected and compact, $f(S^1)$ is also connected and compact and so must be a closed bounded interval $[a,b]$. This is a contradiction since $[a,b]$ is not homeomorphic to S^1 as the first contains a cut point while the latter does not.

(h) Let X be a second countable space with the property that sequences converge to at most one point. Then X is Hausdorff.

True. Let $x, y \in X$ such that $x \neq y$. Let $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ be a countable nested neighbourhood basis at x and y respectively, which exists as X is first countable. Suppose $U_n \cap V_n \neq \emptyset$ for every $n \in \mathbb{N}$. Construct a sequence $(z_n)_{n\in\mathbb{N}}$ by choosing z_n be a point in $U_n \cap V_n$. Then z_n is a sequence that converges to x and y which is a contradiction. Therefore, $U_n \cap V_n$ is empty for some $n \in \mathbb{N}$, and hence there exists disjoint neighbourhoods of x and y. Since x and y were arbitrary distinct points, we conclude that X is Hausdorff.

(i) $\{0, 1\}^{\mathbb{N}}$ is metrizable where $\{0, 1\}$ is equipped with the discrete topology.

True. This follows from the fact that the discrete topology is metrizable and the countable product of metrizable spaces is metrizable

(j) Let $f : X \to Y$ be a surjective continuous function. If X is separable, then Y is too.

True. Let A be a countable dense set in X. Since f is continuous, $Y = f(\overline{A}) \subseteq \overline{f(A)} \subseteq Y$ and so $f(A)$ is a countable dense set in Y.

(k) **(*bonus*** [**2**]**)** A space X has k connected components if and only if there exists a surjective continuous function $f : X \to \{1, 2, ..., k\}$ where $\{1, 2, ..., k\}$ is equipped with the discrete topology.

False. The backward direction doesn't hold since more than one connected component could map to the same number. For example $X = (0, 1) \cup (2, 3)$ has two connected components but the constant function $f \equiv 1$ is a surjective continuous function from X to $\{1\}$.

Problem 7: *bonus* [2]

Let $f : X \to Y$ be a surjective continuous map where X is compact and Y is Hausdorff. Define an equivalence relation on X as follows: $x \sim y$ if $f(x) = f(y)$. Show that $X/\sim$ is homeomorphic to Y.

Since f is constant on equivalence classes, it reduces to a map $\bar{f} : X/\sim \to Y$. By 4b in assignment 3, $\bar{f}$ is continuous. It is also bijective by the definition of the equivalence relation. Since the natural map $\pi : X \to X/\sim$ is continuous and X is compact, it follows that $X/\sim = \pi(X)$ is compact. Since Y is Hausdroff, the closed map lemma from lectures implies that $\bar{f}$ is a homeomorphism.

Note: You can get up to 58/50 in this test.