* Midterm will be June 23 (9 - 11:30) AM

* Next week is the last week of lectures for the first semester

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Countability axioms & separable spaces

First countable is the first countability axiom (discussed last lecture)

Recall countable is the smallest infinity in the sense that for any infinite set X, $\exists f: \mathbb{N} \to X$ s.t. f is injective.

Countable basis is easier to work with than an uncountable basis:

<u>Def:</u> A topological space is second countable if it admits a countable basis.

$\mathbb{R}$ can be "approximated" by a countable set $\mathbb{Q}$ in the sense that $\overline{\mathbb{Q}} = \mathbb{R}$

<u>Def:</u> A topological space is separable if it admits a countable dense subset.

<u>Ex:</u> * $\mathbb{R}$ is second countable since $\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{Q}\}$ is a countable basis

* $\mathbb{R}$ is separable since $\overline{\mathbb{Q}} = \mathbb{R}$

Lemma: $A \subseteq X$ is dense iff every open set U intersects A.

Proof: -

Proposition: Second countable $\Rightarrow$ first countable.

Proof: Let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis.
Let $x \in X$. Then $\mathcal{N}_x = \{B_n \mid n \in \mathbb{N},\ x \in B_n\}$ is a countable neighbd basis of x.

Is every metric space second countable or/and separable?
An uncountable discrete space is not second countable nor separable but is metrizable.

Theorem: Let X be a topological space. If X is second countable, then it's separable. The other way around is true if X is a metric space.

Proof: $(\Rightarrow)$ Let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable basis for X. Pick $x_n \in B_n$ for every $n \in \mathbb{N}$, and let $A = \{x_n \mid n \in \mathbb{N}\}$. Since every open set contains B_n for some $n \in \mathbb{N}$, it follows that every open set intersects A & hence $\overline{A} = X$. So X is separable ☒

$(\Leftarrow)$ Suppose X is a separable metric space. Let $A = \{x_n \mid n \in \mathbb{N}\}$

be a countable dense set.

A

choose countable nbhd basis at every point.

for $a \in A$, define $\mathcal{N}_a = \{ B_{\frac{1}{n}}(a) \mid n \in \mathbb{N} \}$

Then $\mathcal{B} = \bigcup_{a \in A} \mathcal{N}_a$ is a countable basis (show this) (jump)

and hence X is second countable.

Exc: Why does this idea not work for arbitrary spaces? or for first countable spaces?

So for metric spaces, second countable $\iff$ separable.

Ex: $\mathbb{R}_\ell$:

- Hausdorff ✓
- first countable ✓
- Separable? $\overline{\mathbb{Q}} = \mathbb{R}_\ell$ so yes ✓
- second countable? No.

Suppose $\mathcal{B}$ is a countable basis.

For every $x \in \mathbb{R}_\ell$, $\exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq [x, x+1)$

But $B_x \neq B_y$ for $x \neq y$. So $\mathcal{B}$ can't be countable since the map $x \in \mathbb{R}_\ell \longmapsto B_x \in \mathcal{B}$ is injective.

$x < y$

Then $B_y \subseteq [y, y+1)$ and so $x \notin B_y$

But since $x \in B_x$, $B_x \neq B_y$

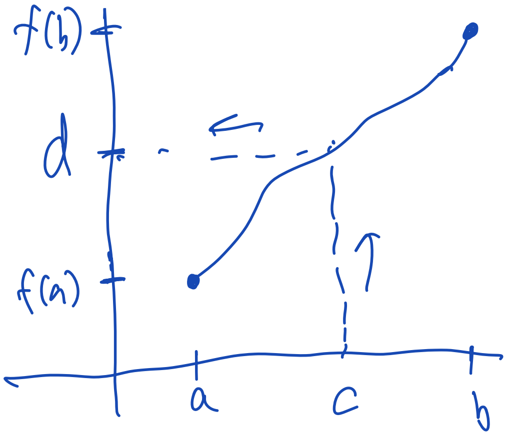
So $\mathbb{R}_\ell$ is separable but not second countable.

$\implies \mathbb{R}_\ell$ is not metrizable.

Connectedness

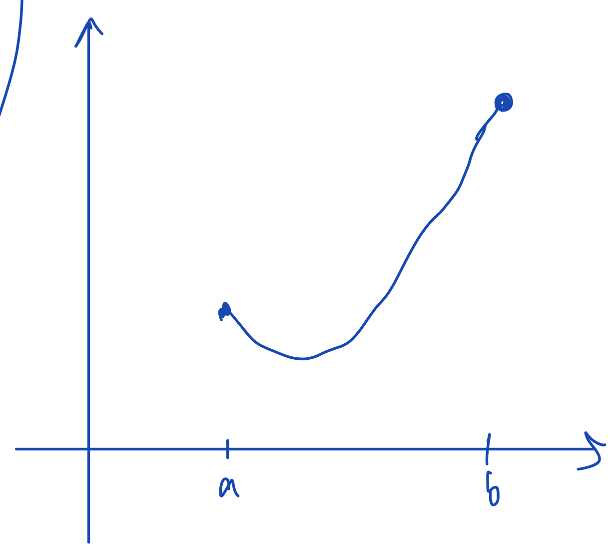
Recall from MAT 457:

IVT:



Doesn't only depend on the continuity of f. It also depends on the topology of $[a,b]$.
(depends on the topological property that will be later called connectedness)

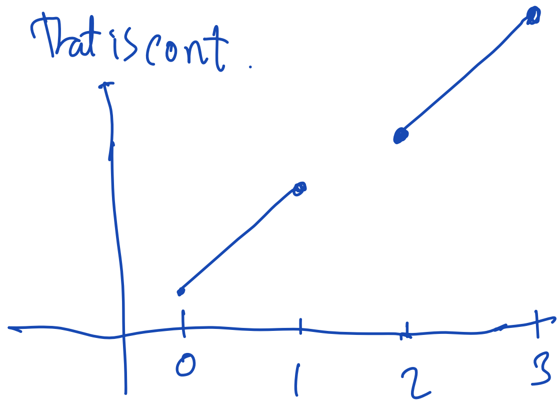
EVT:



Doesn't only depend on the continuity of f. It also depends on the topology of $[a,b]$
(depends on the topological property that will be later called compactness)

Let $f : [0,1] \cup [2,3] \to \mathbb{R}$

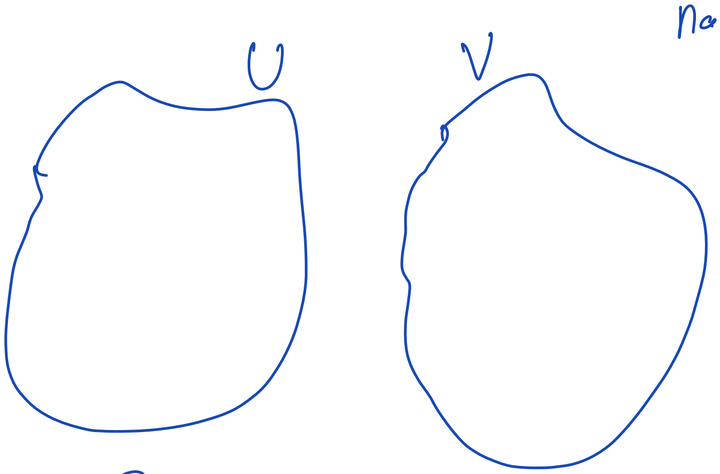
That is cont.



but the conclusion of IVT is false.

The required topological property for IVT is hold will be called connectedness. How will we define it?

Not connected set



$[0,1] \cup [2,3]$

<u>Def:</u> A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

We say X is connected if there doesn't exist a separation of X.

Connectedness is a topological property and is invariant under homeomorphisms.

If X is not connected, then $X = U \cup V$ for some disjoint nonempty open sets U and V. In particular, $U^c = V$ and so U is closed. Similarly $V^c = U$ and so V is closed.

<u>Lemma</u>: X is connected iff the only clopen sets are X and $\emptyset$.

<u>proof</u>: existence of nontrivial clopen set U
$\iff$ existence of a separation (U and U^c)

<u>Ex</u>: * $(X, \mathcal{T}_{discrete})$ is not connected since $X = \{x\} \cup \{x\}^c$
($\{x\}$ is clopen set) open, open

* $(X, \mathcal{T}_{indiscrete})$ is connected (the only open sets are X and $\emptyset$)

(*) $X = [0,1) \cup (1,2]$ equipped with the subspace topology is not connected. $[0,1)$ and $(1,2]$ form a separation of X.

Check Piazza.

(7) The only nonempty connected subspaces of $\mathbb{R}$ are intervals

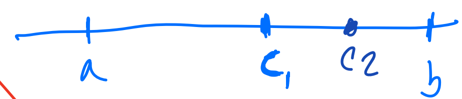
Proof: Let $Y \subseteq \mathbb{R}$ be connected. Wlog, suppose Y is not a singleton.

Let $a, b \in Y$ s.t. $a < b$.

It suffices to show that $[a,b] \subseteq Y$. (show that it indeed suffices)

Let $c_1 = \sup \{ d \in [a,b] \mid [a,d) \subseteq Y \}$

Let $c_2 = \inf \{ d \in [a,b] \mid (d,b] \subseteq Y \}$



Suppose $c_1 < c_2$, Then $U = [a, c_1]$ and $[c_2, b]$ form a separation of $Y \cap [a,b]$, which is a contradiction.

So $c_1 \geq c_2 \Rightarrow Y \cap [a,b] = [a,b]$.

$([a,b] \subseteq Y)$

Theorem: Let X be connected. Let $f: X \to Y$ that is continuous. Then $f(X)$ is connected.

Proof: Suppose $f(X)$ isn't connected. So $\exists$ a separation U and V open nonempty disjoint sets s.t. $U \cup V = f(X)$. (U and V are open in $f(X)$)

So $\exists$ open sets $\tilde{U}$ and $\tilde{V}$ in Y s.t. $U = \tilde{U} \cap f(X)$

and $V = \tilde{V} \cap f(X)$.

Then $f^{-1}(\tilde{U})$ and $f^{-1}(\tilde{V})$ form a separation of X, which is a contradiction. $\blacksquare$

Post-lecture-Practice-Question

#1) Do the exercises above.

#2) We will fill in the gaps in the proof of "Separable metric spaces are second countable."

We claim that $\mathcal{B} := \bigcup_{a \in A} \mathcal{N}_a$ is a countable basis for X, where

$$\mathcal{N}_a := \{ B_{1/n}(a) \mid n \in \mathbb{N} \}.$$

Let U be an open subset of X. Show that for every $x \in U$, $x \in B_{1/n}(a)$ for some $a \in A$ and $n \in \mathbb{N}$. Conclude that $\mathcal{B}$ is a basis for X.

#3) For $q \in \mathbb{Q}$, $\mathcal{N}_q = \{ [q, q+\frac{1}{n}) \mid n \in \mathbb{N} \}$ is a neighbd basis at q wrt the lower limit topology $\mathbb{R}_\ell$.

Consider $\mathcal{B} = \bigcup_{q \in \mathbb{Q}} \mathcal{N}_q$. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let $U = [x, x+1)$. Show that $\nexists\, B \in \mathcal{B}$ s.t. $x \in B \subseteq [x, x+1)$. Conclude that $\mathcal{B}$ is not a basis for $\mathbb{R}_\ell$. Reflect on where the proof in #2 fails when X is first countable but not metrizable.

#4) Show that a subspace of a first countable (second countable) space is also first countable / second countable

#5) Is a subspace of a separable space always separable? What if it's an open subspace?

#6) Show the finite product of first countable / second countable / separable space is first countable / second countable / separable.
Which ones fail for countably infinite products? What about uncountable infinite products?

#7) Let $X = \{ (x,y) \in \mathbb{R}^2 \mid y=0 \text{ or } xy=1 \}$.

Show X is not connected.

#8) Show that if X is a separable, then every collection of disjoint open sets is countable.

#9) Is $\mathbb{R}_\ell$ connected?