$\times$ - ——— ———

Recall: ⊛ finite and countably infinite product of metric space is a metric space.

What about an uncountable product of metric spaces such as $\mathbb{R}^{\mathbb{R}}$?

We want to study properties of metric spaces:

$\boxed{*}$ Metric spaces are Hausdorff

$\boxed{*}$ Metric space are $\underbrace{\text{first countable}}$.

↳ every point admits a neighbd basis.

First countable allows us to construct sequences:

The sequence lemma: Let X be a topological space. Let $A \subseteq X$. If $\exists x_n \in A$ converges to x, then $x \in \bar{A}$. The converse is true if X is first countable.

Proof: $\Rightarrow$ ✓

$(\Leftarrow)$ Let $x \in \bar{A}$. Let $\mathcal{U}_x = \{U_n \mid n \in \mathbb{N}\}$ a countable neighbd basis at x.

$V_n = \bigcap_{i=1}^{n} U_i$, $V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4 \supseteq \cdots$

Since $x \in \bar{A}$, $U_n \cap A \neq \emptyset$ $\forall n \in \mathbb{N}$ and so $\bigcap_{i=1}^{n} U_i \cap A \neq \emptyset$ $\forall n \in \mathbb{N}$.

neighbd of x

Pick $x_n \in \bigcap_{i=1}^{n} U_i \cap A$, which defines a sequence in A.

Let U be an arbitrary neighbd of x. $\exists N \in \mathbb{N}$ s.t. $\bigcap_{i=1}^{N} U_i \subseteq U$

$\Rightarrow$ $x_n \in \bigcap_{i=1}^{n} U_i \subseteq \bigcap_{i=1}^{N} U_i \subseteq U$ $\forall n \geq N$, and so $x_n \to x$.

$\blacksquare$

So all metric spaces satisfy the sequence lemma.

Ex: ⊛ $(\mathbb{R}, \mathcal{T}_{ray})$ is first countable since for $x \in \mathbb{R}$,

$\mathcal{N}_x = \{(x-\frac{1}{n}, \infty) \mid n \in \mathbb{N}\}$ is a countable neighbd basis at x. (So it satisfies the sequence lemma)

but not Hausdorff so not metrizable.

⊛ $\mathbb{R}_\ell$. It's both first countable and Hausdorff.

↳ for $x \in \mathbb{R}$, let

$\mathcal{N}_x = \{(x-\frac{1}{n}, x+\frac{1}{n}) \mid n \in \mathbb{N}\} \cup \{[x, x+\frac{1}{n}) \mid n \in \mathbb{N}\}$

(so $\mathbb{R}_\ell$ satisfies the sequence lemma)

Is $\mathbb{R}_\ell$ metrizable?

⊛ $(\mathbb{R}, \mathcal{T}_{co\text{-}countable})$. Let $A = \mathbb{R} \setminus \{7\}$

Then $7 \in \overline{A}$ but $\nexists$ $x_n \in A$ converging to 7. So doesn't

Satisfy the sequence lemma and hence not first countable.

Def: Let $f: X \to Y$. We say f is sequentially continuous if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Theorem: Let $f: X \to Y$. f is continuous $\Rightarrow$ f is sequentially cont. The converse is true if X is first countable.

Proof: $\Rightarrow$ ✓

($\Leftarrow$) Suppose f is sequentially cont. Let $A \subseteq X$. We want to show $f(\overline{A}) \subseteq \overline{f(A)}$

Let $y \in f(\overline{A})$, Then $y = f(x)$ for some $x \in \overline{A}$.

Since X satisfies the sequence lemma, $\exists x_n \in A$ converging to x.

Since f is sequentially cont, $f(x_n) \to f(x) = y$

$\Rightarrow$ $y \in \overline{f(A)}$

Corollary: If X is a metric space and $f: X \to Y$, then f is cont iff it's sequentially cont.

Is an uncountable product of metric spaces metrizable?

Consider $\mathbb{R}^{\mathbb{R}} = \{ f : \mathbb{R} \to \mathbb{R} \}$

Let $A = \{ f : \mathbb{R} \to \mathbb{R} \mid f(x) = 1$ for all but finitely many $x \in \mathbb{R} \}$

Claim ⊛ $f_0 \equiv 0 \in \overline{A}$:

⊛ $\nexists\ f_n \in A$ converging to f_0
(very trivial once you prove 2b: convergence in $\mathbb{R}^{\mathbb{R}}$ is equivalent to pointwise convergence).

Let $\prod_{\alpha \in \mathbb{R}} U_\alpha$ be a basis neighbd of f_0. So U_α is a neighbd of 0 $\forall \alpha \in \mathbb{R}$ and $U_\alpha = \mathbb{R}$ for all $\alpha \in \mathbb{R}$ except for $\alpha = \alpha_1, \cdots, \alpha_n$.

So consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ for $x \neq \alpha_1, \cdots, \alpha_n$ and $f(x) = 0$ for $x = \alpha_1, \cdots, \alpha_n$. So $f \in \prod_{\alpha \in \mathbb{R}} U_\alpha \cap A$.

$\Rightarrow f_0 \in \overline{A}$

$\Rightarrow$ $\mathbb{R}^{\mathbb{R}}$ doesn't satisfy the sequence lemma and so is not first countable, and not metrizable.

In fact: Uncountable product of first countable nontrivial spaces is not first countable.

$\Rightarrow$ Uncountable product of metric spaces is not metrizable.

Post - Lecture - Practice - Questions

#1) Do the exercises above.

#2) Let $\{X_\alpha\}_{\alpha \in J}$ be an uncountable collection of indiscrete topological spaces. Show that X_α is first countable $\forall \alpha$ and $\prod X_\alpha$ is also first countable. Does that contradict what was proven in tutorial?

#3) a) Show that the sequence $x_n = (-1)^n$ doesn't converge in $(\mathbb{R}, \tau_{co\text{-}finite})$

b) Show that the sequence $x_n = n$ converges to every number in $(\mathbb{R}, \tau_{co\text{-}finite})$

c) What are converging sequences in $(\mathbb{R}, \tau_{co\text{-}finite})$?

d) Is $(\mathbb{R}, \tau_{co\text{-}finite})$ first countable or Hausdorff.

#4) What are the converging sequences in $(\mathbb{R}, \tau_{co\text{-}countable})$? Show that sequences converge to at most 1 point but the space is not Hausdorff.

5) Let (X, d_X) and (Y, d_Y) be metric spaces. Show that $f: X \to Y$ is cont iff $\forall x \in X$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x,y) < \delta$.

Use this to give another proof for the equivalence of continuity & sequential continuity.