* Assignment 2 is posted.

Recall from last lecture:

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

We define two topologies on $\prod_{\alpha \in J} X_\alpha$:

* Box topology generated by sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for every $\alpha \in J$.

* Product topology generated by sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α for every $\alpha \in J$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$.

If J is finite, then $\mathcal{T}_{box} = \mathcal{T}$

In general $\mathcal{T}_{box} \supseteq \mathcal{T}$

What is the difference and why is $\mathcal{T}$ more important?

Similarities:

* Let $A_\alpha \subseteq X_\alpha$ be a subspace. Then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$ (in both $\mathcal{T}$ and $\mathcal{T}_{box}$)

* If X_α is Hausdorff $\forall \alpha \in J$, then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff in both $\mathcal{T}$ and $\mathcal{T}_{box}$.

* Let $A_\alpha \subset X_\alpha$ be a subspace. Then in both $\mathcal{T}$ and $\mathcal{T}_{box}$,

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

↳ Proof: $\overline{A_\alpha} = \bigcap_{\substack{B_\alpha \supseteq A_\alpha \\ B_\alpha \text{ is closed}}} B_\alpha$

Then $\prod \overline{A_\alpha} = \prod \bigcap_{\substack{B_\alpha \supseteq A_\alpha \\ B_\alpha \text{ is closed}}} B_\alpha$

$$= \bigcap_{\substack{\alpha \in J \\ B_\alpha \supseteq A_\alpha \\ B_\alpha \text{ is closed}}} \prod B_\alpha$$

$\left(\prod B_\alpha\right)^c = \bigcup_{\alpha \in J} \prod_{\beta \in J} U_{\alpha\beta}$ where $U_{\alpha\beta} = X_\beta$ for $\alpha \neq \beta$ and $U_{\alpha\beta} = B_\alpha^c$ for $\alpha = \beta$

↳ is open in both $\mathcal{T}$ and $\mathcal{T}_{box}$

$\Rightarrow$ $\prod B_\alpha$ is closed in both $\mathcal{T}$ and $\mathcal{T}_{box}$

So $\prod \overline{A_\alpha} \supseteq \overline{\prod A_\alpha}$ since $\overline{\prod A_\alpha}$ is a larger intersection.

For the other direction, start with $x = (x_\alpha)_{\alpha \in J} \in \prod \overline{A_\alpha}$

$\Rightarrow \forall \alpha \in J$, Every neighbd U_α of x_α intersects A_α

$\Rightarrow$ Every box topology basis neighbd $\prod U_\alpha$ of $(x_\alpha)_{\alpha\in J}$ intersects $\prod A_\alpha$
(and so in particular, same holds for every product topology basis neighbd)

$\Rightarrow$ $(x_\alpha)_{\alpha\in J} \in \overline{\prod A_\alpha}$ wrt both $\mathcal{T}$ and $\mathcal{T}_{box}$.

$\blacksquare$

Differences between $\mathcal{T}_{box}$ and $\mathcal{T}$:

Theorem 1: let $f: A \to \prod_{\alpha\in J} X_\alpha$ be given by

$$f(x) = (f_\alpha(x))_{\alpha\in J} \quad \text{where } f_\alpha : A \to X_\alpha$$

Let $\prod X_\alpha$ be equipped with the product topology.
Then f is continuous $\Longleftrightarrow$ f_α is continuous $\forall \alpha\in J$.

Proof: We observe that

f is cont $\Longleftrightarrow$ $f^{-1}(\prod U_\alpha)$ is open whenever U_α is open in X_α $\forall \alpha\in J$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha\in J$.

$\Longleftrightarrow$ $\bigcap_{\alpha\in J} f_\alpha^{-1}(U_\alpha)$ is open whenever U_α is open in X_α $\forall \alpha\in J$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in J$.

$\Longleftrightarrow$ for every choice of finitely many $\alpha \in J$, say $\alpha_1, \dots, \alpha_n$, and for every open set U_{α_i} in X_{α_i} for $1 \leq i \leq n$, $\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$ is open.

($\Longrightarrow$) Suppose f is continuous. This implies for every $\alpha \in J$, $f_\alpha^{-1}(U_\alpha)$ is open for every open set U_α.

$\Longrightarrow$ f_α are continuous for all $\alpha \in J$

fails for box top.

($\Longleftarrow$) Suppose f_α is continuous $\forall \alpha \in J$.
Then for every $\alpha_1, \dots, \alpha_n \in J$ and every sets U_{α_i} open in X_{α_i}, we have that $\bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i})$ is open and so f is continuous.

$\boxtimes$

Note that ($\Longleftarrow$) might fail since arbitrary intersections of open sets is not necessarily open.

<u>Theorem 2</u>: Let x_n be a sequence in $\prod_{\alpha \in J} X_\alpha$ equipped with the product topology.

Then $x_n \to x \iff \pi_\alpha(x_n) \to \pi_\alpha(x)$
$\forall \alpha \in J$.

(2a in Assignment 2).

<u>Ex:</u> $\mathbb{R}^{\mathbb{N}} = \{ x : \mathbb{N} \to \mathbb{R} \} = \{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \}$
$= \{ \text{sequences in } \mathbb{R} \}$

Let $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$
$: t \mapsto (t, t, t, \cdots)$

so $f(t) = (f_\alpha(t))_{\alpha \in \mathbb{N}}$ where $f_\alpha : \mathbb{R} \to \mathbb{R}$
$: t \longmapsto t$

f is cont when $\mathbb{R}^{\mathbb{N}}$ is equipped with the product top using Thm 1 (since all f_α are continuous.)

But : $f^{-1}\left(\prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) \right) = \{0\}$

open in box topology

not open in $\mathbb{R}$

So f is not continuous if $\mathbb{R}^{\mathbb{N}}$ is equipped with box topology.
(so Thm 1 is false for box topology).

Also let x_n be a sequence in $\mathbb{R}^{\mathbb{N}}$ defined by

$$x_n = \left\{\frac{1}{nm}\right\}_{m \in \mathbb{N}}$$

In 2b in Assignment 2, you will prove that

$x_n \to \bar{0}$ in the $\mathcal{T}$ but not in $\mathcal{T}$ box.

zero sequence

This shows that Thm 2 is false in box topology.

Metric & Product Topology

When is the product topology metrizable?

Let (X, d) be a metric space.

Define $\bar{d} : X \times X \to [0, \infty)$ by

$$\bar{d}(x, y) := \min\{d(x, y), 1\}$$

This is called the standard bounded metric.

<u>Theorem</u>: $\bar{d}$ induces the same topology as d.

(Proven in Assignment 2 & the book)

Let $(X_1, d_1), \dots, (X_n, d_n)$ be metric spaces.

Is the Product topology on $\prod_{i=1}^{n} X_i$ metrizable?

Guess: Define $d(x,y) := \max_{1\le i\le n} d_i(x_i, y_i)$ for $x,y \in \prod_{i=1}^{n} X_i$

which is motivated by the square metric on $\mathbb{R}^n$.

Theorem: d is a metric on $\prod_{i=1}^{n} X_i$ that induces the Product topology.

(i.e. finite Product of metric spaces is a metric space)

What about an infinite Product of metric spaces?

Let us first consider $\mathbb{R}^{\mathbb{N}}$

Guess: (*) $d(x,y) := \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$ ← motivated from the euclidean distance in $\mathbb{R}^n$.

Not well-defined.

(*) $d(x,y) := \sup_{i\in\mathbb{N}} |x_i - y_i|$. Again Not well defined.

(*) $d(x,y) := \sup_{i\in\mathbb{N}} \{ \bar{d}(x_i, y_i) \}$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$ defined by $\bar{d}(x,y) := \min\{|x-y|, 1\}$ for $x,y \in \mathbb{R}$.

This is indeed a metric on $\mathbb{R}^{//}$

Def: Let J be an index set. We define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by $\bar{\rho}(x,y) := \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$

$$:= \sup_{\alpha \in J} \bar{d}(x_\alpha, y_\alpha)$$

where $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha \in J}$ are points in $\mathbb{R}^J$.
and $\bar{d}$ is the standard bounded metric on $\mathbb{R}$.
We call $\bar{\rho}$ the uniform metric on $\mathbb{R}^J$, and the topology it induces is called the uniform topology.

Theorem: $\mathcal{T}_{prod} \subseteq \mathcal{T}_{unif} \subseteq \mathcal{T}_{box}$
These three topologies are distinct when J is infinite.

Proof: Let $x \in \mathbb{R}^J$.
let $\prod_{\alpha \in J} U_\alpha$ be a product topology basis set containing x.

Then $U_\alpha = \mathbb{R}$ $\forall \alpha \in J$ except say $\alpha_1, \cdots, \alpha_n$.

Let $\varepsilon > 0$ small enough s.t. $(x_{\alpha_i} - \varepsilon, x_{\alpha_i} + \varepsilon) \subseteq U_{\alpha_i}$ for $1 \le i \le n$

radius

$\Rightarrow B_{\bar\rho}(x, \varepsilon) \subseteq \prod_{\alpha \in J} U_\alpha$

metric, center

$\Rightarrow \mathcal{T}_{prod} \subseteq \mathcal{T}_{unif}$

Show $\mathcal{T}_{unif} \subseteq \mathcal{T}_{box}$ (Show for any $x \in \mathbb{R}^J$ and $r > 0$, $\prod_{\alpha \in J} (x_\alpha - \frac{r}{2}, x_\alpha + \frac{r}{2}) \subseteq B_{\bar\rho}(x, r)$)

Show they are distinct if J is infinite:

(*) find an open set in $\mathcal{T}_{box}$ That is not in $\mathcal{T}_{unif}$.
(try a product of intervals with radii going to 0).

(*) Find an open set in $\mathcal{T}_{unif}$ That is not in $\mathcal{T}_{prod}$
(are balls wrt $\bar\rho$ open in $\mathcal{T}_{prod}$?)

$\blacksquare$

So The uniform metric was not a correct guess since it induced a topology distinct from both $\mathcal{T}_{prod}$ and $\mathcal{T}_{box}$.

Let us make another guess.

We want a metric s.t. The open balls include all real numbers far enough in The sequence.

So let $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.
We want $\pi_n(B_r(x)) = \mathbb{R}$ for n big enough.

Guess: for $x, y \in \mathbb{R}^{\mathbb{N}}$, define

$$D(x,y) = \sup\left\{ \frac{\overbrace{\bar{d}(x_i - y_i)}^{\leq 1}}{i} \;\middle|\; i \in \mathbb{N} \right\}$$

let $r > 0$. for $n > \frac{1}{r}$,

we have that $\frac{\bar{d}(y,0)}{n} \leq \frac{1}{n} < r$

and so for n big enough, $\pi_n(B_r(\bar{0})) = \mathbb{R}$
(for all $n > \frac{1}{r}$)

Theorem: D is a metric on $\mathbb{R}^{\mathbb{N}}$ that induces the product topology.

Proof: first, we show D is a metric.

- $D(x,y) = 0 \iff x = y$
- $D(x,y) = D(y,x) \quad \forall x, y \in \mathbb{R}^{\mathbb{N}}$
- Let $x, y, z \in \mathbb{R}^{\mathbb{N}}$.

Then $\frac{\bar{d}(x_i, z_i)}{i} \leq \frac{\bar{d}(x_i, y_i)}{i} + \frac{\bar{d}(y_i, z_i)}{i}$

$\leq D(x,y) + D(y,z)$

Since the above holds $\forall i \in \mathbb{N}$,

$$D(x,z) \leq D(x,y) + D(y,z).$$

Show that D induces the product topology

(broken into steps in post-lecture-practice-question)

So indeed $\mathbb{R}^{\mathbb{N}}$ is metrizable !!

Theorem: Let J be finite or countable.
Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of metric spaces.
Then $\prod_{\alpha \in J} X_\alpha$ is metrizable.

Proof: J finite is done ✓
J countable: define a metric D as we did for $\mathbb{R}^{\mathbb{N}}$.

Enumerate $J = \{\alpha_1, \alpha_2, \cdots\}$
Let d_{α_i} be the metric on X_{α_i}.
Let $\bar{d}_{\alpha_i}$ be the standard bounded metric on X_{α_i} inducing the same topology as d_{α_i}.

Then define $D(x,y) = \sup\left\{ \frac{\overline{d_{\alpha_i}}(x_i, y_i)}{i} \;\middle|\; i \in \mathbb{N} \right\}$

and show D induces the product topology on $\prod_{\alpha \in J} X_\alpha$.

$\blacksquare$

What about an uncountable infinite product?

So is $\mathbb{R}^{\mathbb{R}}$ metrizable?

We need some tools/techniques to quickly see if a topology is metrizable. We already know one:

* If $(X, \mathcal{T})$ is not Hausdorff then X is not metrizable.

Def: Let $(X, \mathcal{T})$ be a topology. Let $x \in X$.

A neighbd basis at x is a collection $\mathcal{N}_x$ of neighbd of x with the property that for every neighbd U of x, $\exists N \in \mathcal{N}_x$ s.t. $N \subseteq U$.

Def: $(X, \mathcal{T})$ is first countable if every point $x \in X$ admits a countable neighbd basis.

<u>Proposition</u>: Every metric space is first countable.

<u>Proof</u>: Let $x \in X$. Then $\mathcal{N}_x := \{ B_{1/n}(x) \mid n \in \mathbb{N} \}$ is a countable neighbd basis since for any neighbd U of x, $\exists\, r > 0$ s.t. $B_r(x) \subseteq U$ since the collection of balls is a basis for X.
Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < r$ and so $B_{1/n}(x) \subseteq B_r(x) \subseteq U$ $\square$

Post-Lecture-Practice-Questions.

1) Do the exercises above.
2) Let $A \subseteq X$ where X is a metric space.
   Show the subspace topology on A is induced by $d|_A$.
3) Show that $A \subseteq \prod X_\alpha$ is closed in the Product ~~box~~ topology implies that it's closed in the box ~~Product~~ topology.
4) Let (X_i, d_i) be metric spaces for $1 \le i \le n$.
   Show that $d(x,y) := \sqrt{\sum_{i=1}^{n} d_i(x_i,y_i)^2}$ for $x,y \in \prod_{i=1}^{n} X_i$ is a metric that induces the product (& box topology) on $\prod_{i=1}^{n} X_i$.

5) We will show that D defined in lectures induces the product topology on $\mathbb{R}^{\mathbb{N}}$.

Let $\prod_{i=1}^{\infty} U_i$ be a product topology basis set where $U_i = \mathbb{R}$ $\forall i \in \mathbb{N}$ except for $i = i_1, \cdots, i_k$. Let $x \in \prod_{i=1}^{\infty} U_i$. Choose $\varepsilon > 0$ small enough s.t. $(x_{i_j} - \varepsilon, x_{i_j} + \varepsilon) \subseteq U_{i_j}$ for $1 \leq j \leq n$.

a) Show that $B_D(x, \varepsilon) \subseteq \prod_{i=1}^{\infty} U_i$. This shows that the metric topology is finer than the product topology.

b) Now let $x \in \mathbb{R}^{\mathbb{N}}$ and $r > 0$. Show that $\exists N \in \mathbb{N}$ s.t. $D(x,y) = \sup\{\bar{d}(x_i, y_i) \mid 1 \leq i \leq N\}$ $\forall y \in B_D(x, r)$.

(i.e. $\pi_n(B_D(x,r)) = \mathbb{R}$ $\forall n > N$).

c) Show that $\prod_{i=1}^{\infty} U_i \subseteq B_D(x,r)$ where $U_i = (x_i - \frac{r}{2}, x_i + \frac{r}{2})$ for $1 \leq i \leq N$ and $U_i = \mathbb{R}$ for $i > N$. This shows that the product topology is finer than the metric topology.

6) Since $(X, \mathcal{T}_{discrete})$, we know X is first countable. In fact, we can find a finite neighbhd basis at each point. Show this.

7) #6 in Munkres Section 20.

8) #4 in Munkres Section 20.

9) Find an infinite product of topological space in which $\mathcal{T}_{box} = \mathcal{T}_{prod}$.