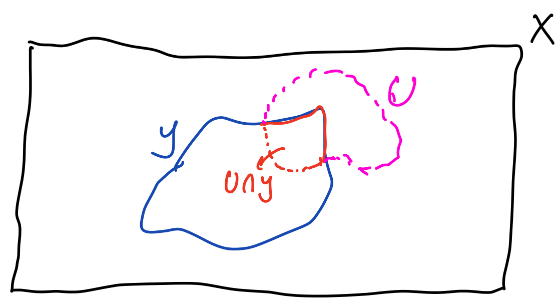
* Assignment 2 will be posted soon.

Recall:

Let X be a topological space. Let $Y \subseteq X$.



Theorem: a) $A \subseteq Y$ is closed in Y $\iff$ $A = B \cap Y$ for some set $B \subseteq X$ that is closed in X.

b) The closure of $A \subseteq Y$ in Y $= \bar{A} \cap Y$ where $\bar{A}$ is the closure of A in X.

Proof

Let X and Y be topological space.

Let $A \subseteq X$ and $B \subseteq Y$. Then what topology does $A \times B$ have?

$$A \times B$$

Subspace topology from $X \times Y$ equipped with Product topology $=$ Product topology where A and B are equipped with the Subspace topology.

Theorem: Equip A and B with the supspace topoloy.
Let τ_1 be the Product topology on $A \times B$.
Equip $X \times Y$ with the Product topology.
Let τ_2 be the Subspace topology on $A \times B \subseteq X \times Y$.
Then $\tau_1 = \tau_2$

Proof: τ_1 is generated by the basis

$$\left\{ U_A \times V_B \;\middle|\; U_A \subseteq A \text{ and } V_B \subseteq B \text{ are open wrt the subspace top on } A \text{ and } B \right\}$$

$$= \left\{ U_A \times V_B \;\middle|\; U_A = U \cap A \text{ and } V_B = V \cap B \text{ where } U \text{ and } V \text{ are open in } X \text{ and } Y \right\}$$

$$= \left\{ U \times V \cap A \times B \;\middle|\; U \text{ is open in } X \text{ and } V \text{ is open in } Y \right\}$$

↳ also generates τ_2

$\blacksquare$

Maps into Products: Let $f: A \to X \times Y$ be a function.

We can write $f = (f_1, f_2)$ defined by $a \mapsto (f_1(a), f_2(a))$ where $f_1: A \to X$ and $f_2: A \to Y$.

f_1 and f_2 are called the coordinate functions of f.

Theorem: f is cont $\iff$ f_1 and f_2 are cont.

Proof: f is cont $\iff$ $f^{-1}(U \times V)$ is open whenever U and V are open.

$\iff$ $f_1^{-1}(U) \cap f_2^{-1}(V)$ is open whenever U and V are open.

So that proves $(\Leftarrow)$

for $(\Rightarrow)$: assume f is cont. Chose $V = Y$, $U \subseteq X$ an arbitrary open set.

Then $f_1^{-1}(U) \cap f_2^{-1}(Y)$ is open

$\Rightarrow$ $f_1^{-1}(U)$ is open

$\Rightarrow$ f_1 is cont & similarly by choosing $U = X$, we get that f_2 is cont.

∎

Remark: $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$

You can also prove the above Thm using the above identity & the fact that the composition of cont functions is cont.

Infinite Product Spaces

Let X be a topological space. Let J be an index set.
A J-tuple of element of X is a map $x: J \to X$.
We denote $x(\alpha)$ by x_α, and J-tuple x by $(x_\alpha)_{\alpha \in J}$.

We define X^J := The set of all J-tuples of X

If J is finite, we can choose it to be $J = \{1, 2, \cdots, n\}$
Then $X^J = \underbrace{X \times X \times X \cdots \times X}_{n \text{ times}} = X^n$

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets.

The cartesian product of $\{A_\alpha\}_{\alpha \in J}$ is defined as follows:

$$\prod_{\alpha \in J} A_\alpha := \left\{ x : J \to \bigcup_{\alpha \in J} A_\alpha \;\middle|\; \begin{array}{c} x_\alpha := x(\alpha) \in A_\alpha \\ \forall \alpha \in J \end{array} \right\}$$

denoted by $(x_\alpha)_{\alpha \in J}$

We simplify the notation by the dropping the index set.

$\prod A_\alpha$, (x_α)

Note that $X := A_\alpha = A_\beta \quad \forall \alpha, \beta \in J$, then $\prod_{\alpha \in J} A_\alpha = X^J$

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces.

We wish to define a topology on $\prod_{\alpha \in J} X_\alpha$.

As before, we directly check $\mathcal{B}_{box} := \{ \prod U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open} \}$ make a basis for a topology $\mathcal{T}_{box}$ called the box topology.

Wrt this topology, we directly observe that the projection maps $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ are continuous.

We can also equip $\prod_{\alpha \in J} X_\alpha$ with the coarsest topology in which all projection maps are cont.

The topology must contain $\mathcal{C} := \{ \pi_\beta^{-1}(U_\beta) \mid \beta \in J,\ U_\beta \subseteq X_\beta \text{ is open} \}$

Since finite intersections of open sets are open, then

The topology must also contain $\mathcal{B} := \{$ finite intersection of sets in $C \}$ which forms a basis for a topology $\mathcal{T}$ which we call The Product topology.

Sets in $\mathcal{B}$ are of the form $\prod_{\alpha \in J} U_\alpha$ where $U_\alpha \subseteq X_\alpha$ are open and $U_\alpha = X_\alpha$ except for finitely many.

Clearly $\mathcal{B}_{box} \supseteq \mathcal{B} \Rightarrow \mathcal{T}_{box} \supseteq \mathcal{T}$
so the box topology is finer than the product topology.

If J is finite, then $\mathcal{T}_{box} = \mathcal{T}$

* $\mathcal{T}$ is the default topology on a cartesian product.

Post-lecture-Practice-Questions

1) Do the exercises above.
2) Show the composition of continuous functions are cont.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.
Let $g : \mathbb{R} \to \mathbb{R}^2$ be defined by $g(x) := (x, f(x))$.
Show g is cont. Hint: Use one of the theorems in this lecture

4) Let $f : X \to Y$ be a homeomorhism. Let $A \subseteq X$.
Show $f|_A : A \to f(A)$ is a homeomorphism.

5) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\prod_{\alpha \in J} X_\alpha$
Show that $(x_n)_{n \in \mathbb{N}} \to x$
iff $\pi_\alpha(x_n) \to \pi_\alpha(x)$ $\forall \alpha \in J$.

6) Show $\overline{A \times B} = \overline{A} \times \overline{B}$
$\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$

7) What is $\mathbb{R}^{\mathbb{N}}$? Describe the box & product topology
Show they are different by finding an set $U \subseteq \mathbb{R}^{\mathbb{N}}$
that is open in $\mathcal{T}_{box}$ but not in $\mathcal{T}$.
Is $\mathbb{R}^{\mathbb{N}}$ metrizable with $\mathcal{T}_{box}$? $\mathcal{T}$?

8) Let $C(\mathbb{R})$ be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Convince yourself that $C(\mathbb{R}) \subseteq \mathbb{R}^{\mathbb{R}}$
Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $C(\mathbb{R})$ that

converges uniformly to $f \in C(\mathbb{R})$. Does $f_n \to f$ wrt subspace topology inherited from $\mathbb{R}^{\mathbb{R}}$?

9) Let $X_n = \{1, 2\}$ with the discrete topology.
Let $X = \prod_{n \in \mathbb{N}} X_n$. Find explicitly the $\mathcal{T}_{box}$ and $\mathcal{T}$ and show they aren't equal.

10) Let $X = (\mathbb{R}, \mathcal{T}_{discrete})$

Is the product topology on $X^{\mathbb{N}}$ the discrete topology? What about the box topology on $X^{\mathbb{N}}$.

11) Let $A_\alpha \subseteq X_\alpha$ be a closed set for each $\alpha \in J$.

Is $\prod_{\alpha \in J} A_\alpha$ closed in the product top? box topol.?