* Assignment is due today at 5pm.

Let X be a topological space.

A Theorem from last lecture:

(1) $\overline{A\cup B} = \bar{A}\cup\bar{B}$

Proof: $A\cup B \subseteq \bar{A}\cup\bar{B} \Rightarrow \overline{A\cup B} \subseteq \bar{A}\cup\bar{B}$

$A\subseteq A\cup B,\ B\subseteq A\cup B \Rightarrow \bar{A}\subseteq\overline{A\cup B},\ \bar{B}\subseteq\overline{A\cup B} \Rightarrow \bar{A}\cup\bar{B}\subseteq\overline{A\cup B}$

(2) $\bigcup_{\alpha\in I}\bar{A}_\alpha \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}$ (I is an index set)

Proof: $A_{\alpha'} \subseteq \bigcup_{\alpha\in I} A_\alpha \quad \forall \alpha'\in I$

$\Rightarrow \bar{A}_{\alpha'} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha} \quad \forall\alpha'\in I$

$\Rightarrow \bigcup_{\alpha\in I}\bar{A}_\alpha \subseteq \bigcup_{\alpha\in I} A_\alpha$

Attempting the other direction:

$\bigcup_{\alpha\in I} A_\alpha \subseteq \bigcup_{\alpha\in I}\bar{A}_\alpha$

$\Rightarrow \overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I}\bar{A}_\alpha}$

$\uparrow$ $\neq \bigcup_{\alpha\in I}\bar{A}_\alpha$

mistake in last lecture

(3) $\underset{\phi}{\overline{A\cap B}} \overset{\subseteq}{\cancel{=}} \underset{\{0\}}{\bar{A}\cap\bar{B}}$

$A=(-\infty,0)$
$B=(0,\infty)$

Proof: $A\cap B\subseteq\bar{A}\cap\bar{B}$

$\Rightarrow \overline{A\cap B}\subseteq\bar{A}\cap\bar{B}$

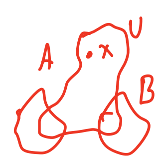
other direction?

Let $x\in\bar{A}\cap\bar{B}$

$\Rightarrow$ Every neighborhood intersects A and B.

$\cancel{\Rightarrow}$ Every neighborhood intersects $A\cap B$

$\Rightarrow x\in\overline{A\cap B}$



(4) $\overline{\bigcap_{\alpha\in I} A_\alpha} \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$ mistake in last lecture. I incorrectly wrote "=" instead of "$\subseteq$".

Proof: $\bigcap_{\alpha\in I} A_\alpha \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$ since $A_\alpha \subseteq \overline{A_\alpha}$ $\forall \alpha\in I$

$\Rightarrow \overline{\bigcap_{\alpha\in I} A_\alpha} \subseteq \overline{\bigcap_{\alpha\in I} \overline{A_\alpha}} = \bigcap_{\alpha\in I} \overline{A_\alpha}$

Theorem: Let $f:(X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$. The following are equivalent:

(a) f is continuous

(b) $f^{-1}(A)$ is closed in X whenever A is closed in Y

(c) $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$.

Proof: (a) $\Leftrightarrow$ (b) due to $\circledast$ A is closed $\Leftrightarrow$ A^c is open

$\circledast$ $f^{-1}(A^c) = f^{-1}(A)^c$

(a) $\Rightarrow$ (c) Let f be continuous. Let $A \subseteq X$

Let $y \in f(\overline{A}) \Rightarrow y = f(x)$ for some $x \in \overline{A}$

Let U_y be an arbitrary neighbhd of y. Then $f^{-1}(U_y) \ni x$ is open since f is cont. And so $f^{-1}(U_y)$ intersects A since $x \in \overline{A}$

$\Rightarrow$ U_y intersects $f(A)$

Since U_y was an arbitrary neighbhd of y, $y \in \overline{f(A)}$ ∎

$(c) \Rightarrow (b)$ Let $B \subseteq Y$ be a closed set.

WTS $A := f^{-1}(B)$ is closed. Recall the following two facts:

$*$ $f(f^{-1}(B)) \subseteq B$ for any $B \subseteq Y$

$\#$ $f^{-1}(f(A)) \supseteq A$ for any $A \subseteq X$

$$f(A) = f(f^{-1}(B)) \overset{\text{by } *}{\subseteq} B$$

$\Rightarrow$ $\overline{f(A)} \subseteq B$ $\Rightarrow$ $f(\overline{A}) \subseteq B$ since (c) is satisfied.

Also $\overline{A} \underset{\text{by } \#}{\subseteq} f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A$

$\therefore$ $\overline{A} \subseteq A \Rightarrow A$ is closed.

∎

Can we add (d): $f(x_n) \to f(x)$ whenever $x_n \to x$

This is called sequential continuity and is not always equivalent to continuity.

<u>Related</u>: $x \in \overline{A}$ $\cancel{\Leftarrow\!\!\Rightarrow}$ $\exists x_n \in A$ s.t. $x_n \to x$

Proposition: Let $f: (X, \tau_X) \to (Y, \tau_Y)$ be a homeomorphism. Then X and Y as topological spaces are the "same" up to renaming of the elements x "=" $f(x)$ (indistinguishable as topological spaces) in the sense that every topological property is invariant under f:

(a) A is open / closed $\iff$ $f(A)$ is open / closed

(b) x is a limit / isolated / boundary / interior point of A $\iff$ $f(x)$ is a limit / isolated / boundary / interior of $f(A)$

(c) $x_n \to x \iff f(x_n) \to f(x)$

(d) X is Hausdorff $\iff$ Y is Hausdorff

(e) $A \subseteq X$ is compact / connected $\iff$ $f(A)$ is compact / connected

Remark: Be careful, nontopological properties / concepts depending on the metric will not necessarily be invariant under homeomorphisms. (for ex, $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $f(x) = \tan x$ is a homeomorphism but doesn't preserve boundedness)

Remark: Topological properties are also called topological invariants due to the above proposition.

New spaces from old: Product Topology

Let (X, τ_X) & (Y, τ_Y) be topological spaces.
The Product topology on $X \times Y$ is the topology generated by the basis $\{U \times V \mid U \in \tau_X, V \in \tau_Y\}$ verify this is a basis

So $A \subseteq X \times Y$ is open iff $\forall (x,y) \in A$, $\exists$ neighbd U_x of x and a neighbd V_y of y s.t. $U_x \times V_y \subseteq A$.

Lemma: If $\mathcal{B}_X$ is a basis for τ_X and $\mathcal{B}_Y$ is a basis for τ_Y, then $\{B_X \times B_Y \mid B_X \in \mathcal{B}_X \text{ and } B_Y \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.

Proof

Ex: The product topology on $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ is generated by

$$\{ (a,b)\times(c,d) \mid a<b,\ c<d \}$$

$$= \{ \text{open balls wrt the metric } d((x_1,y_1),(x_2,y_2)) := \max\{|x_1-x_2|, |y_2-y_1|\} \}$$

↳ generates the standard topology on $\mathbb{R}^2$.

The product topology on $\mathbb{R}^2$ = The standard topology on $\mathbb{R}^2$

The projection functions: $\pi_1 : X\times Y \to X$
$: (x,y) \mapsto x$

and $\pi_2 : X\times Y \to Y$
$: (x,y) \mapsto y$

Note π_1 and π_2 are surjective.

If $U \subseteq X$ is open, then $\pi_1^{-1}(U) = U\times Y$ is open in $X\times Y$

If $V \subseteq Y$ is open, then $\pi_2^{-1}(V) = X\times V$ is open in $X\times Y$

$\Rightarrow$ π_1 and π_2 are continuous.

We can ask: What is the coarsest / ~~finest~~ discrete topology on $X\times Y$ s.t. both π_1 and π_2 are cont?

The coarsest topology must contain $C := \{\pi_1^{-1}(U) \mid U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) \mid V \in \mathcal{T}_Y\}$

$$= \{U \times Y \mid U \in \mathcal{T}_X\} \cup \{X \times V \mid V \in \mathcal{T}_Y\}$$

Define $\mathcal{B} = \{\text{finite intersections of sets in } C\}$ makes a basis for a topology on $X \times Y$ which would be the coarsest topology containing C.

Note that $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$ which is a basis for the product topology.

since $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$

<u>Theorem</u>: The coarsest topology on $X \times Y$ in which π_1 and π_2 are continuous is the product topology.

Add the details to complete the proof

<u>Remark</u>: The above Thm is false for infinite products.

New space from old: Subspace topology

Let $(X, \mathcal{T})$ be a topological space.
Let $Y \subseteq X$. Is there a topology on Y that is "inherited" from X?

Let's first think of metric spaces:
Let (X, d) be a metric space. Let $Y \subseteq X$.
Then d restricts to a metric on Y:
$d_Y : Y \times Y \to [0, \infty)$ defined by $d_Y(x,y) = d(x,y)$ for $x, y \in Y$.

(notion of distance carries over directly)

for $x \in Y$, $r > 0$,
the open balls $B_r^Y(x) = \{ y \in Y \mid d_Y(x,y) < r \}$
$= B_r(x) \cap Y$

The topology on Y generated by the open balls $B_r^Y(x)$, $x \in Y$, $r > 0$
is $\mathcal{T}_Y = \{ U \cap Y \mid U \subseteq X \text{ is open in } X \}$ Show

Since $A \subseteq Y$ is open $\iff$ A is a union of open balls $B_r^Y(x)$ $\substack{x \in Y \\ r > 0}$
$\iff$ A is a union of sets of the form $B_r(x) \cap Y$ $\substack{x \in Y \\ r > 0}$

$\iff A = U \cap Y$ for some set U that is open in X.

will not be unique

So $A \subseteq Y$ is open in Y $\iff$ $A = U \cap Y$ for some set U that is open in X.

Inspired by the above, we define the subspace in general topological spaces:

Let $(X, \mathcal{T})$ be any topological space
Let $Y \subseteq X$

Then the subspace topology $\mathcal{T}_Y$ on Y is the topology it inherits from X and is defined by $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$

show $\mathcal{T}_Y$ is indeed a topology.

Lemma: If $\mathcal{B}$ is a basis for X, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ makes a basis for the subspace topology.

Proof

Example: $Y := (0,1] \subseteq \mathbb{R}$

The subspace topology is generated by the basis $\{(a,b) \cap (0,1] \mid a < b\}$

So $(\frac{1}{2}, 1]$ is open in Y since $(\frac{1}{2}, 1] = (\frac{1}{2}, 2) \cap Y$
$= (\frac{1}{2}, 3) \cap Y$

but $(\frac{1}{2}, 1]$ is not open in X.

Remark: Let $A \subseteq Y$. When we talk about a topological property for A, we need to specify which topology

A is open / closed in Y $\neq$ A is open / closed in X

$x \in Y$ is a limit point of A $\neq$ $x \in Y$ is a limit point of X

$\vdots$

Note that Y is always open in Y but not necessarily open in X.

Lemma: If $Y \subseteq X$ is open in X, then

$A \subseteq Y$ is open in Y $\Leftrightarrow$ A is open in X.

The subspace topology $\mathcal{T}_Y = \{U \in \mathcal{T} \mid U \subseteq Y\}$

Show this

Post-Lecture-Practice-Questions.

1) Do the exercises above.

2) Find a collection $A_\alpha \subseteq \mathbb{R}$, $\alpha \in I$ satisfying

$$\bigcup \overline{A_\alpha} \nsubseteq \overline{\bigcup A_\alpha}$$

3) Find a collection $A_\alpha \subseteq \mathbb{R}$, $\alpha \in I$ satisfying

$$\bigcap_{\alpha\in I} \overline{A_\alpha} \nsubseteq \overline{\bigcap_{\alpha \in I} A_\alpha}$$

4) a) Let $f: X \to Y$ be a cont function between Topological spaces. Let $x_n \to x$. Show $f(x_n) \to f(x)$.

b) Let $f: X \to Y$ be a function with the property that $f(x_n) \to f(x)$ whenever $x_n \to x$.

i) Suppose X and Y are metric spaces. Show that f is cont.

ii) Find an example of X, Y and $f: X \to Y$ s.t. f is not continous but has the above property. (Hard)

5) Show that the subspace topology on $\mathbb{Z} \subseteq \mathbb{R}$ is the discrete topology.

6) Show that the subspace topology of $Y \subseteq X$ is a metric topology whenever X is a metric space. (Any subspace of a metric space is a metric space)

7) Any subspace of a Hausdorff space is Hausdorff.
The product of Hausdorff spaces is Hausdorff.

8) Let $Y \subseteq X$. Show that the inclusion map $i : Y \to X$ defined by $i(x) = x$ for $x \in Y$ is continuous.
Show that the subspace topology is the coarsest topology in which i is continuous.

9) a) Show that X is Hausdorff iff the diagonal $\Delta := \{(x,x) \in X \times X\}$ is closed in $X \times X$.

b) Show that Δ is homeomorphic to X where Δ is equipped with the subspace topology it inherits from the product topology on $X \times X$.