* Tutorials & OH today

* Mistake when explaining liminf / limsup

Let x_n be a sequence in $\mathbb{R}$. Then

* $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \boxed{\inf_{k\geq n} x_k} = \sup_{n\in\mathbb{N}} \inf_{k\geq n} x_k$ $\quad =: y_n$

* $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \boxed{\sup_{k\geq n} x_k} = \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$ $\quad =: z_n$

Define $y_n := \inf_{k\geq n} x_k$ which is $\nearrow$ and $\longrightarrow \liminf x_n$

$z_n := \sup_{k\geq n} x_k$ which is $\searrow$ and $\longrightarrow \limsup x_n$

Also $y_n \leq x_n \leq z_n \quad \forall n \in \mathbb{N}$

$\Rightarrow \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ and $\lim_{n\to\infty} x_n$ exists iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

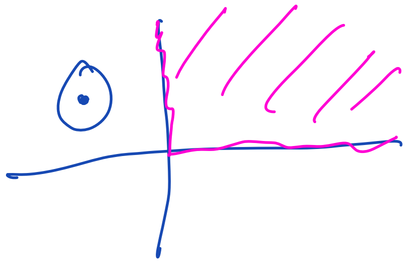
Closed Sets, Closure, Interior

Let X be a topological space.

<u>Def</u>: $A \subseteq X$ is closed if A^c is open.

Ex:

- $[a,b]$ is closed since $[a,b]^c = (-\infty, a) \cup (b, \infty)$ which is open
- on $\mathbb{R}^2$, define $A := \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$



A is closed since

$A^c = (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$ is open

- Can sets both open and closed? Yes! X and $\emptyset$ are always open & closed. (We call this clopen)

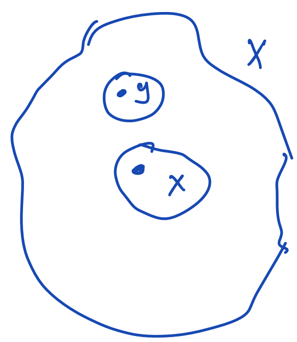
Can sets be neither open nor closed? yes! $[0,5)$

- $\{x\}$ are these always closed? no.

Proposition: Let X be a Hausdorff space. Then singletons are closed.

Proof: Let $x \in X$. For $y \in X \setminus \{x\}$, we can choose a neighbd U_y of y that doesn't contain x.

Then $\{x\}^c = \bigcup_{\substack{y \in X \\ y \neq x}} U_y$ which is open.



Theorem:
(1) $\emptyset$ and X are closed
(2) Arbitrary intersection of closed sets is closed.
(3) finite union of closed sets is closed.

Def: Let $A \subseteq X$. The closure of A, denoted by $\overline{A}$, is the intersection of all closed sets containing A.
The interior of A, denoted by $\mathrm{Int}(A)$, is the union of all open sets contained in A.

It directly follows:

1) $\mathrm{Int}(A) \subseteq A \subseteq \overline{A}$
2) $\mathrm{Int}(A)$ is open and $\overline{A}$ is closed
3) $\mathrm{Int}(A)$ is the "largest" open set contained in A in the sense that if $U \subseteq A$ is open, then $U \subseteq \mathrm{Int}(A)$.
4) $\overline{A}$ is the "smallest" closed set containing A in the sense that if $B \supseteq A$ is closed, then $\overline{A} \subseteq B$.
5) $\mathrm{Int}(A) = A \iff A$ is open
$\overline{A} = A \iff A$ is closed

Theorem: (1) $x \in \overline{A} \iff$ every neighbhd of x intersects A.
(2) Let $\mathcal{B}$ be a basis. $x \in \overline{A} \iff$ every basis neighbhd of x intersects A.

Proof: $x \notin \overline{A} \iff \exists$ closed set $C \supseteq A$ s.t. $x \notin C$
$\iff \exists$ closed set $C \supseteq A$ s.t. $\underbrace{x \in C^c}_{\text{open}}$

$\Longleftrightarrow$ $\exists$ neighd of x that doesn't intersect A. $\square$

Ex: 1) $\overline{(0,1]} = [0,1]$

$\mathrm{Int}(0,1] = (0,1)$

2) $A := \{\frac{1}{n} \mid n \in \mathbb{N}\}$ $\quad \overline{A} = A \cup \{0\}$

3) let $x \in \mathbb{R}^n$, $r > 0$. $\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|x-y\| \leq r\}$

$=: C_r(x)$

In general metric spaces,

is it true that $\overline{B_r(x)} = C_r(x)$? No.

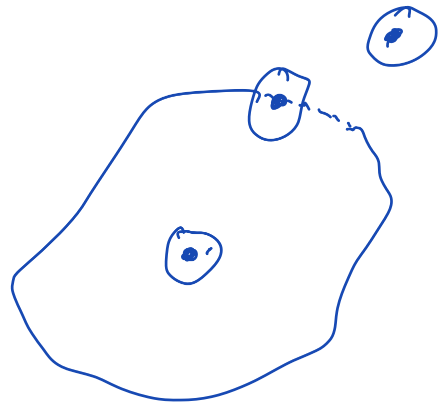
4) $\overline{\mathbb{Q}} = \mathbb{R}$ (every open interval contains a rational #)

Def: $A \subseteq X$ is dense if $\overline{A} = X$.

Limit points, Interior points, boundary points, isolated points.

Def: let $A \subseteq X$. $x \in X$ is a limit point of A if every neighbd of x intersects $A \setminus \{x\}$

$A' := \{\text{limit point of } A\}$



<u>Def</u>: Let $A \subseteq X$. $x \in X$ is a boundary point of A if every neighbd of x intersects both A and A^c.

$$\partial A := \{\text{boundary points}\}$$

$$\partial A \underset{\not\subseteq}{\overset{\not\supseteq}{\neq}} A'$$

Let $A \subseteq X$.

<u>Def</u>: $x \in A$ is an interior point of A if $\exists$ neighbd of x that is contained in A.

<u>Def</u>: Let $A \subseteq X$. $x \in A$ is an isolated point of A if $\exists$ neighbd U of x s.t. $A \cap U = \{x\}$

<u>Theorem</u>: Let $A \subseteq X$

1) $\overline{A} = A \cup A' = A \cup \partial A$

If there are no isolated points of A, then $\overline{A} = A'$

2) $\text{Int}(A) = \{ \text{interior points} \}$

Proof: follows from the equivalent definition of $\overline{A}$ above & def of $\text{Int}(A)$.

Corollary: A is closed iff $A' \subseteq A$
iff $\partial A \subseteq A$

Theorem: (1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

$$\overline{\bigcup_{\alpha \in I} A_\alpha} \overset{?}{\underset{\supseteq}{\subseteq}} \bigcup \overline{A_\alpha} \; ?$$

(2) $$\overline{\bigcap_{\alpha \in I} A_\alpha} \overset{\subseteq}{\cancel{=}} \bigcap_{\alpha \in I} \overline{A_\alpha}$$

(mistake during lecture)

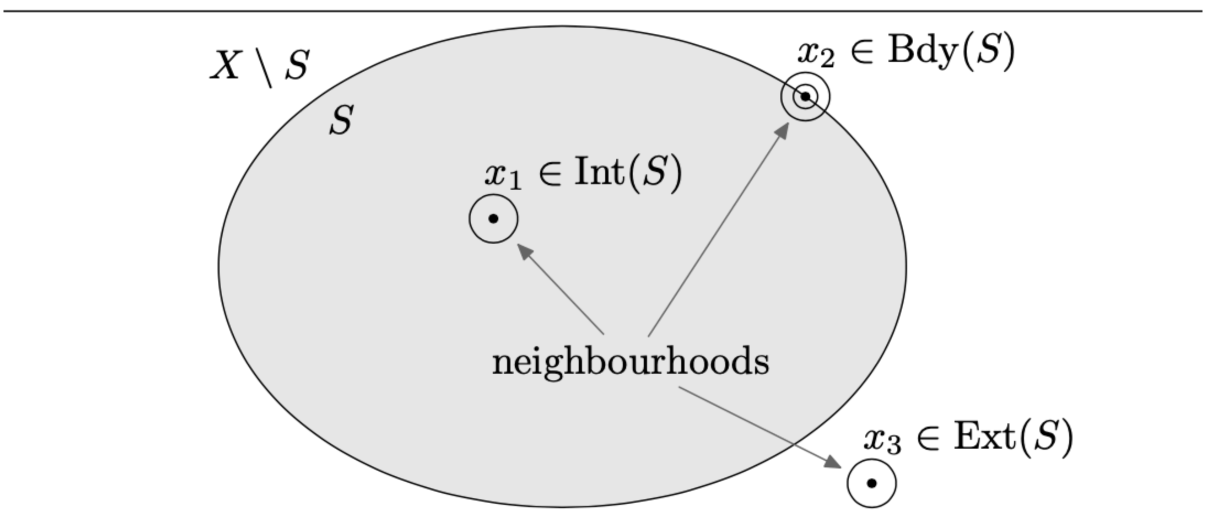
(3) $\overline{A^c} = \text{Int}(A)^c$, $\overline{A}^c = \text{Int}(A^c)$

(4) Decomposition Theorem: Let $A \subseteq X$:

Then X can be decomposed or written as the following disjoint union:

$$X = \underbrace{\text{Int}(A) \cup \rlap{\overbrace{\phantom{\partial A \cup \text{Int}(A^c)}}^{\overline{A^c}}}\partial A}_{\overline{A}} \cup \text{Int}(A^c)$$

Called Exterior of A



Interior, boundary and exterior of a set

Post-Lecture-Practice-Questions

1) Do the exercises above.
2) Show $\overline{\overline{A}} = \overline{A}$ and $\text{Int}(\text{Int}(A)) = \text{Int}(A)$
3) Show ∂A is closed.

4) Is the following true: a) $\overline{A} = \text{Int}(A) \cup A'$
   b) $\overline{A} = \text{Int}(A) \cup \partial A$

5) Show that every point in X is an isolated point for every set $A \subseteq X$ in the discrete topology.

6) Suppose $x_n \to x$. Is it always true that
$\overline{\{x_n \mid n \in \mathbb{N}\}} = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$?

7) Find $\overline{\{7\}}$ in the ray topology. Conclude that $(\mathbb{R}, \tau_{ray})$ is not Hausdorff.

8) Find an example of a collection of sets A_α s.t.

$$\overline{\bigcup_{\alpha \in I} A_\alpha} \neq \bigcup_{\alpha \in I} \overline{A_\alpha}$$

(notice that the right side is not necessarily closed but the left side is)

9) Do #7 in Munkres section 17.

10) Find $\text{Int}(A)$, $\overline{A}$, ∂A, A', isolated points of A for:

a) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{standard})$

b) $(0,1]$ in $\mathbb{R}_\ell$

c) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{ray})$

d) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{co\text{-}finite})$

e) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{co\text{-}countable})$

f) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{discrete})$

g) $(0,1]$ in $(\mathbb{R}, \mathcal{T}_{indiscrete})$

h) $\{(x,y,z) \in \mathbb{R}^3 \mid x=0\}$ in $(\mathbb{R}^3, \mathcal{T}_{standard})$

i) $\mathbb{Q}$ in $(\mathbb{R}, \mathcal{T}_{standard})$