* Assign. is "due on Sunday at 11:59" BUT
* Piazza ☺
* Assign. Mistake in Problem 4. (I removed the question).
* Most Assign. questions need today's lectures (except 1a, 2, 3a)
* OH

Recall:

* Topology = { open sets }
  - ↳ Topological properties / concepts
* Basis $\mathcal{B} \subseteq \mathcal{T}$ ⟶ Topology = { unions of basis sets } (↳ sets in $\mathcal{B}$)

U is open $\iff \forall x \in U, \exists B \in \mathcal{B}$ s.t. $x \in B \subseteq U$

$\iff U$ is a union of basis sets

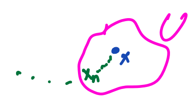
* Topology $\mathcal{T}$ ⟶ $\mathcal{B} \subset \mathcal{T}$ is a basis iff every open set is a union of basis sets.

Topological Properties / Concepts

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B}$ be a basis for $\mathcal{T}$.

<u>Important Terminology:</u> Let $x \in X$. Let U be an open set containing x. We say that "U is a neighbd of x".

<u>Def</u>: Let x_n be a sequence in X. We say x_n converges to x if for every neighbhd U of x, $\exists N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$.



<u>Ex</u>:

* $(X, \mathcal{T}_{discrete})$ $x_n \to x$ iff x_n is eventually constant & equal to x. Show this.
* $(X, \mathcal{T}_{indiscrete})$ All sequences converge to every element. Show this. Hint: Let x_n be a sequence. Let $x \in X$. Then X is the only neighbd of x.
* Define $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\}$. Show this a topology. Show $x_n \to x \iff x \leq \liminf_{n \to \infty} x_n$

$$\left(\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{k \geq n} x_k \right) \quad \lim_{n \to \infty} x_n = x \iff \limsup_{n \to \infty} x_n$$

$\left(\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k \right) \Bigg\} \begin{array}{l} = \liminf_{n\to\infty} x_n \\ = x \end{array}$

Lemma: $x_n \longrightarrow x \iff$ for every basis set B containing x, $\exists N \in \mathbb{N}$ s.t. $x_n \in B$.

Proof: $(\Longrightarrow)$ by definition & by $\mathcal{B} \subseteq \mathcal{T}$.

$(\Longleftarrow)$

Def: Suppose $(X, \mathcal{T}_X)$ & $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is Continuous if $f^{-1}(U)$ is open in X whenever U is open in Y.

$\left[\text{for a set } U \subseteq Y,\ f^{-1}(U) := \{ x \in X \mid f(x) \in U \} \right]$

Preimage

Lemma: Let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Then $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous iff $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}_Y$.

Proof: $(\Longrightarrow)$ follows from $\mathcal{B}_Y \subseteq \mathcal{T}_Y$

$(\Longleftarrow)$

Hint: $f^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha f^{-1}(B_\alpha)$

Def: $f:(X,\tau_X)\to(Y,\tau_Y)$ is a homeomorphism if f is bijective and continuous with continuous inverse.

Def: $f:(X,\tau_X)\to(Y,\tau_Y)$ is an open map if $f(U)$ is open in Y whenever U is open in X.

Lemma: f is a homeomorphism iff it's bijective continuous and $\underbrace{\text{an open map}}_{\iff f^{-1} \text{ is continuous.}}$.

Ex: * What are the continuous functions from $(X,\tau_{discrete})$ to itself? Every function!

* What are the continuous functions from $(X,\tau_{indiscrete})$ to itself? Every function!

* What are the continuous functions from $(X,\tau_{co\text{-}finite})$ to itself? (Think about it)

Metric Topology

A notion of distance between points (called a metric) $\Longrightarrow$ A notion of closeness (called metric topology)

Let X be a set.

<u>Def</u>: A **metric** is a function $d: X \times X \to [0,\infty)$ satisfying:

basic properties we want a sensible notion of distance between points to satisfy

(a) $d(x,y) = 0 \iff x = y$

(b) $d(x,y) = d(y,x) \quad \forall x,y \in X$

(c) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$

Generalization of triangle inequality

The pair (X, d) is called a **metric space**.

<u>Ex</u>:

(*) Let $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$ be a norm on $\mathbb{R}^n$, then $d(x,y) := \|x-y\|$ is a metric. Verify this

(*) Let X be a set. Define $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

Verify d is a metric

This is a metric called the discrete metric.

(*) $X = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$

Then $d(f,g) := \int_a^b |f(x) - g(x)|\, dx$ is a metric on X. Verify d is a metric

(*) Let (X, d) be a metric space.
Then $d'(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$ is another metric.

Def: Let (X,d) be a metric space. For $x \in X$ and $r > 0$, we define the open ball centered at x with radius r as $B_r(x) := \{ y \in X \mid d(x,y) < r \}$

Then $\mathcal{B} := \{ B_r(x) \mid r > 0, x \in X \}$ is a basis for a topology called the metric topology (Verify)

U is open $\iff \forall x \in U$, $\exists$ open ball centered at x inside U
$\iff U$ is a union of open balls.

Terminology: A metric "induces" a topology called metric topology

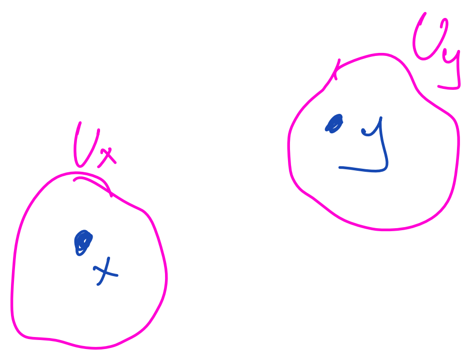
A metric space comes with notions/concepts that are not topological.

- The diameter of set $A \subseteq X$ is defined as
$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x,y)$$

- A is bounded $\operatorname{diam}(A) < \infty$ and unbounded otherwise.
- Cauchy sequences & completeness (we will talk about later).

A metric topology is a special kind of topology. For instance:

$\forall x,y \in X, x \neq y$, $\exists$ neighbd U_x of x and a neighbd U_y of y that are disjoint.



(let $r = \dfrac{d(x,y)}{2}$, then let $U_x = B_r(x)$ and $U_y = B_r(y)$)

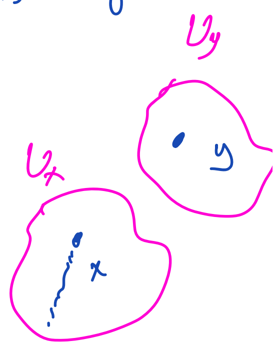
A metric topology can "separate" points but not every topology can (such as the indiscrete topology and the ray topology)

Def: Let (X, τ) be a topological space. (X, τ) is called Hausdorff if $\forall x,y \in X, x \neq y$, there exists a neighbd of x and a neighbd of y that are disjoint.

Metric Spaces are Hausdorff.

Proposition: Let $(X, \mathcal{T})$ be a Hausdorff space. Then sequence converge to at most one point.

Proof: Let x_n be a sequence that converges to x and y. Suppose $x \neq y$. Let U_x and U_y be a neighbd of x and a neighbd of y that are disjoint.



Since $x_n \to x$, $\exists N \in \mathbb{N}$ s.t. $x_n \in U_x$. This in particular implies that U_y contains at most finitely many of the x_n's as U_y is disjoint from U_x. This contradicts that $x_n \to y$.

$\boxtimes$

Corollary: In Metric spaces, sequences converge to at most one point.

Def: A topological space $(X, \mathcal{T})$ is metrizable if $\exists$ metric on X s.t. the metric topology induced by d coincides with $\mathcal{T}$.

$\Longrightarrow$ Indiscrete & Ray topology are not metrizable.

Closed sets, Closure, Interior

Let (X, τ) be a topological space.

Def: A set $A \subseteq X$ is closed if A^c is open.

Theorem: The following properties of closed sets are satisfied:

(1) ϕ and X are closed.

(2) Arbitrary intersection of closed sets is closed

(3) finite union of closed sets is closed.

Proof: follows from properties of open sets together with De Morgan's law.

Def Let $A \subseteq X$. The closure of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

It follows that: * $\bar{A}$ is the smallest closed set containing A in the sense if $B \supseteq A$ is closed, then $B \supseteq \bar{A}$.

* $\bar{A} = A$ iff A is closed.

(Verify)

Post-Lecture-Practice-Questions

1) Do the above exercises.

2) Suppose $f: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ is a homeomorphism. Show $\mathcal{T}_1 = \mathcal{T}_2$.

3) Show that $f: (X, d_1) \to (Y, d_2)$ is continuous iff $\forall x \in X$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_2(f(y), f(x)) < \varepsilon$ whenever $d_1(x, y) < \delta$.

4) a) Let (X, d) be a metric space. Define $d'(x,y) := \frac{d(x,y)}{1+d(x,y)}$. Show d' is a metric.

b) Show that X is bounded wrt d'.

c) Show that $d'(x,y) \leq d(x,y)$

Conclude that $f : (X, d) \to (X, d')$ defined by $f(x) = x$ is continuous.

5) Let $f : X \to Y$ be a map.

For each choice of topology on X, what is the finest topology on Y so that f is continuous?
For each choice of topology on Y, what is the coarsest topology on X so that f is continuous?

6) Is it true that $f : X \to Y$ is continuous iff $\forall x \in X$ and every sequence x_n converging to x, $f(x_n)$ converges to $f(x)$?

7) Let (X, d) be a metric space.
Define $d'(x,y) = \min\{d(x,y), 1\}$.
Show that d' is a metric that induces the same topology as d.