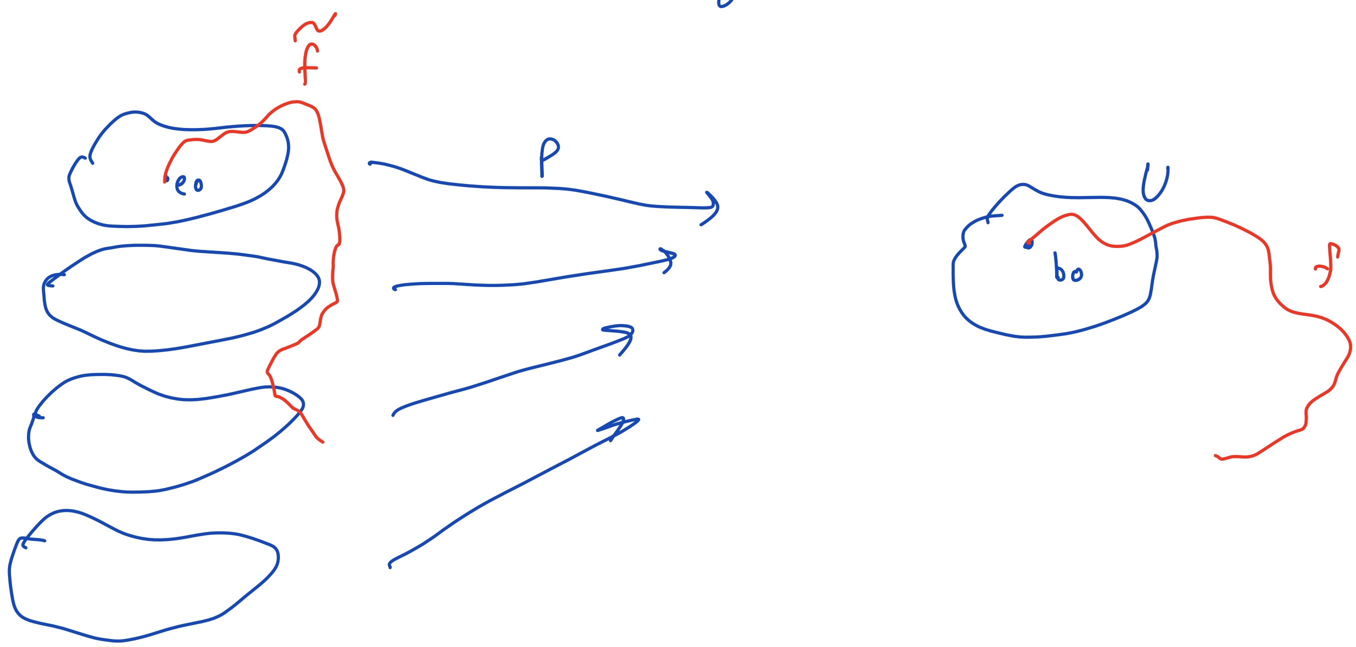
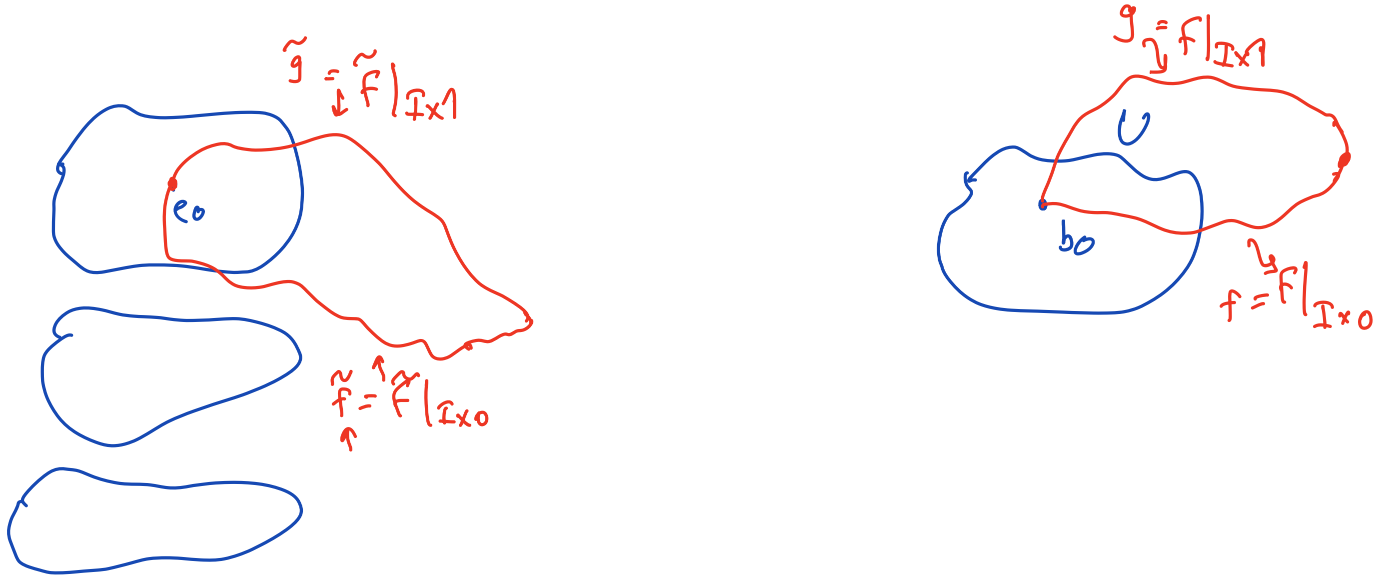
* Course evaluations
* Ass 4 & Ass 5 marking will be done soon.

Recall: Suppose $p: E \to B$ is a covering map and $p(e_0) = b_0$.



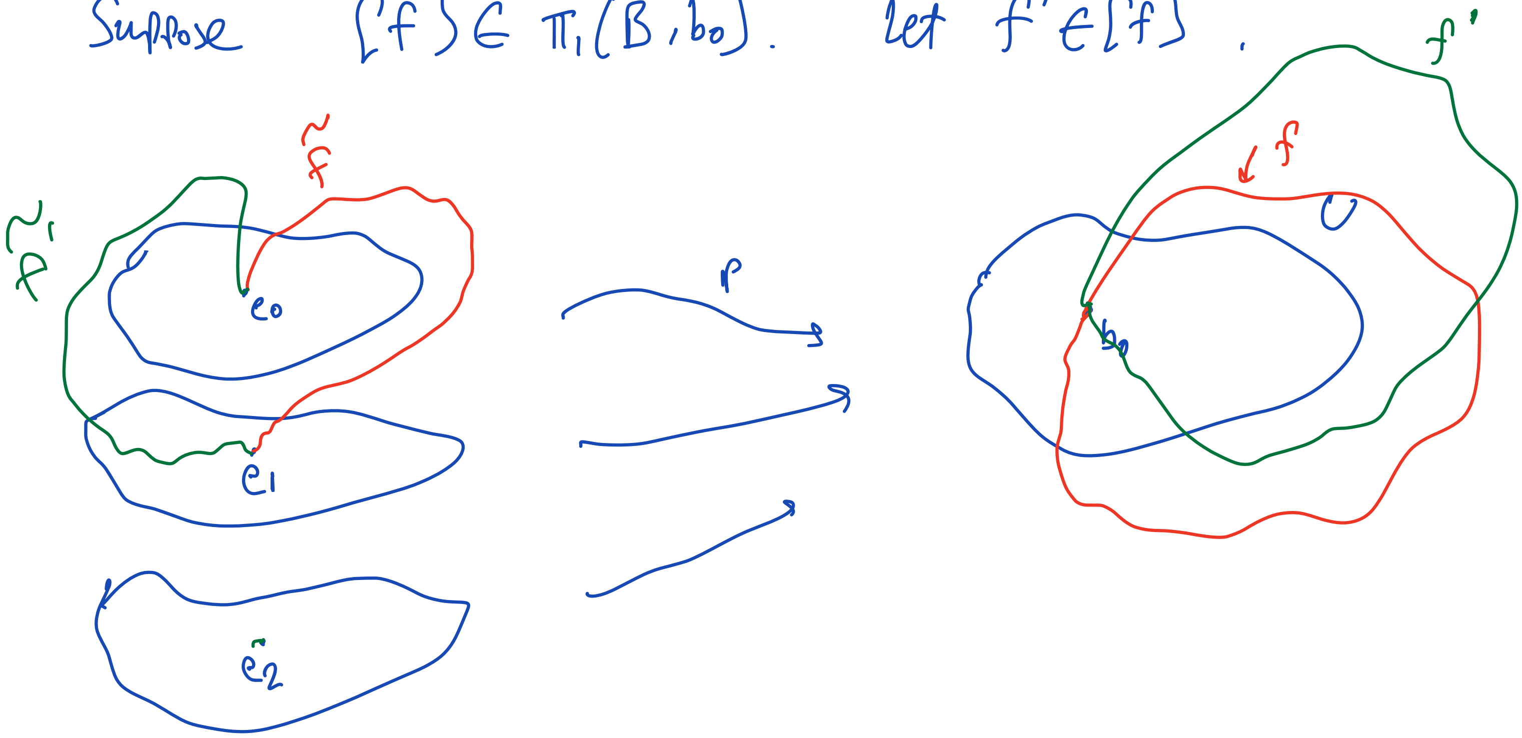
1) A path $f: I \to B$ starting at b_0 can be uniquely lifted to a path $\tilde{f}: I \to E$ starting at e_0.

2) A path homotopy $F: I \times I \to B$ s.t. $F(0,0) = b_0$ can be uniquely lifted to a path homotopy $\tilde{F}: I \times I \to E$ s.t. $\tilde{F}(0,0) = e_0$.



3) If f and g are homotopic paths in B starting at b_0, then the unique lifting $\tilde{f}$ and $\tilde{g}$ (in E), starting at e_0 are also homotopic.

Suppose $[f] \in \pi_1(B, b_0)$. Let $f' \in [f]$.



If $f \simeq_p f'$, then $e_1 := \tilde{f}(1) = \tilde{f}'(1)$ by #3 above

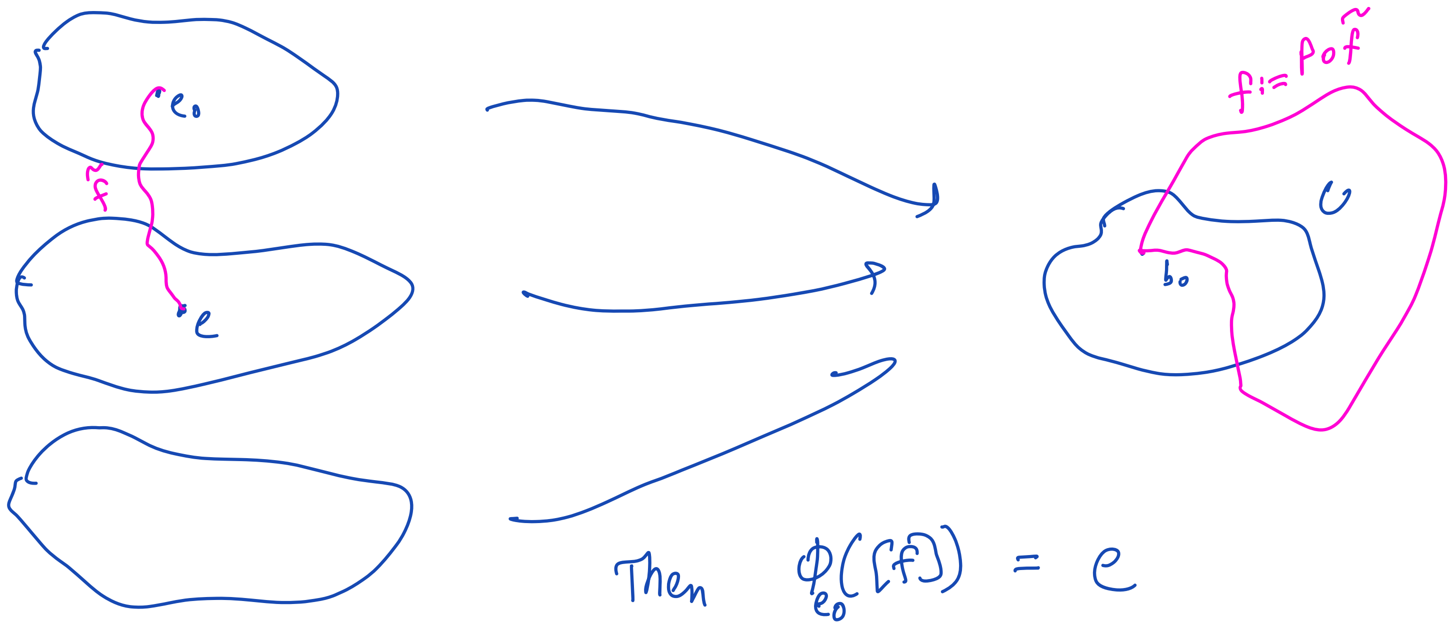
We define a map, $\Phi_{e_0} : \pi_1(B, b_0) \to p^{-1}(b_0)$

by $\Phi_{e_0}([f]) = \tilde{f}(1)$ where $\tilde{f}$ is the unique lifting of f starting at e_0. By the above observations, this map is well defined.

Φ_{e_0} is called the lifting correspondence derived from the covering map p.

Suppose E is path connected. Let $e \in p^{-1}(b_0)$

$\exists$ path $\tilde{f} : I \to E$ starting at e_0 and ending in e.

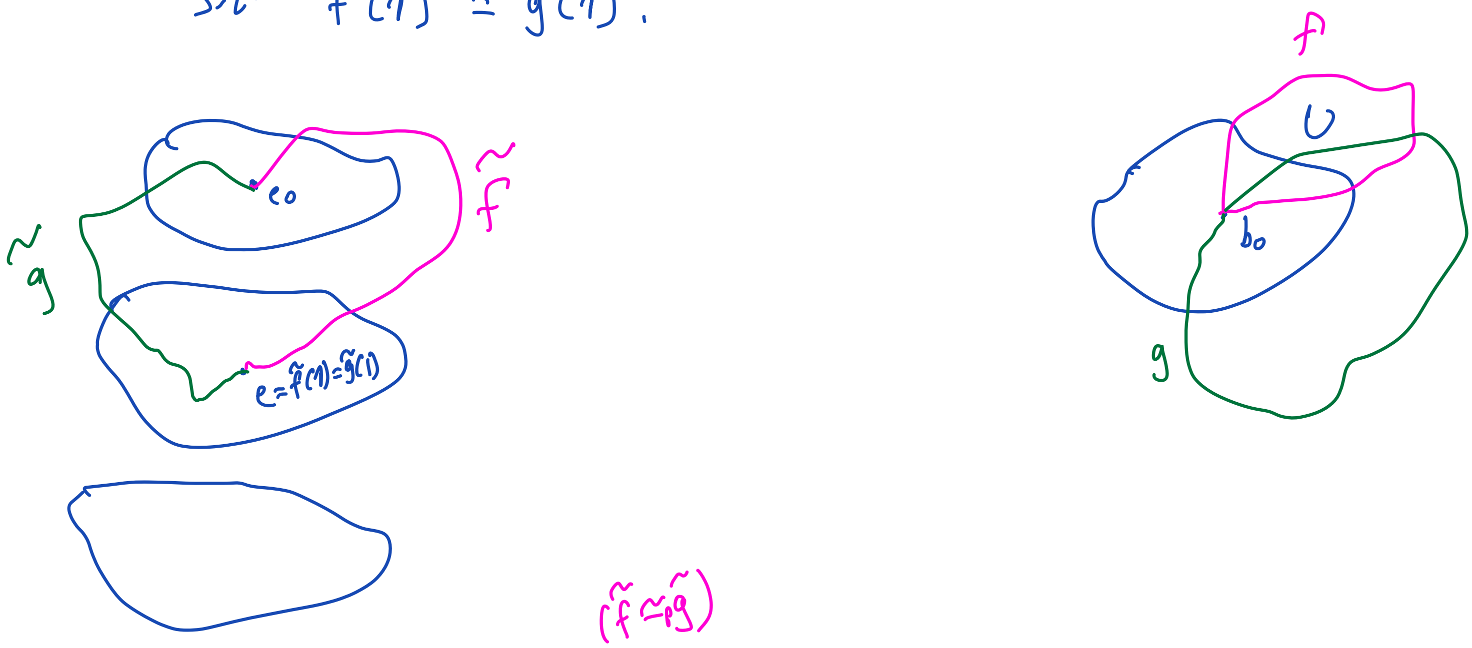


Then $\Phi_{e_0}([f]) = e$

$\Rightarrow$ Φ_{e_0} is surjective

Suppose E is simply connected. Suppose f, g are loops based at b_0

s.t. $\tilde{f}(1) = \tilde{g}(1)$.



$(\tilde{f} \simeq_p \tilde{g})$

Since E_0 is simply connected, $\exists$ path homotopy $\tilde{F} : I \times I \to E$ between $\tilde{f}$ and $\tilde{g}$. Then $F := p \circ \tilde{F} : I \times I \to B$ is a path homotopy between f and g.

Then $f \simeq_p g$. $\Rightarrow$ Φ_{e_0} is injective.

We have the theorem:

Theorem: Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence Φ_{e_0} is surjective. If E is simply connected, then Φ_{e_0} is bijective.

Theorem: The fundamental group of S^1 is isomorphic to $(\mathbb{Z}, +)$.

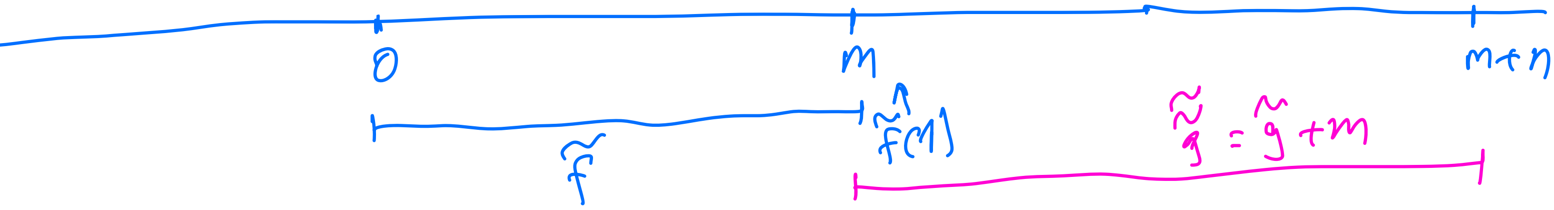
Proof: Let $p: \mathbb{R} \to S^1$ be the covering map defined by $p(x) = (\cos(2\pi x), \sin(2\pi x))$.

Since $\mathbb{R}$ is simply connected, the lifting correspondence $\Phi_0 : \pi_1(S^1, (1,0)) \longrightarrow \underbrace{p^{-1}(1,0)}_{\mathbb{Z}}$ is bijective.

It suffices to show that Φ_0 is a homomorphism.

So we need to show that for every $[f], [g] \in \pi_1(S^1, (1,0))$, $\Phi_0([f] * [g]) = \Phi_0([f]) + \Phi_0([g])$.

Let $[f], [g] \in \pi_1(S^1, (1,0))$. let $m := \Phi_0([f])$ and $n := \Phi_0([g])$



Define $\tilde{\tilde{g}} : I \to \mathbb{R}$ by $\tilde{\tilde{g}}(x) = \tilde{g}(x) + m$

Then
$$
\begin{aligned}
p \circ (\tilde{f} * \tilde{\tilde{g}}) &= (p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}) \\
&= f * (p \circ (\tilde{g} + m)) \\
&= f * (p \circ \tilde{g}) \\
&= f * g
\end{aligned}
$$
$\Rightarrow$ $\tilde{f} * \tilde{\tilde{g}}$ is a lifting of $f * g$.

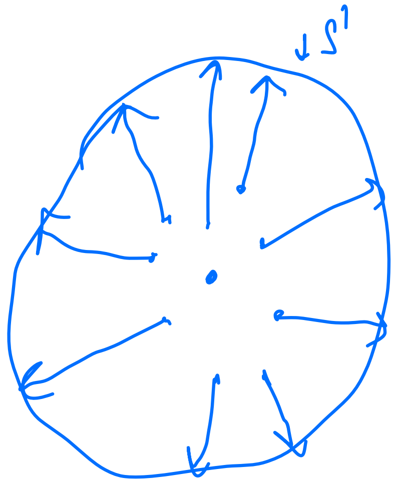
$$
\begin{aligned}
\Rightarrow \quad \Phi_0([f] * [g]) &= \Phi_0([f * g]) \\
&= \tilde{f} * \tilde{\tilde{g}}(1) = m + n \\
&= \Phi_0([f]) + \Phi_0([g])
\end{aligned}
$$
as needed ∎

$\therefore \boxed{\pi_1(S^1, (1,0)) = \mathbb{Z}}$

<u>Recall</u>: A retraction of X onto $A \subseteq X$ is a continuous map $f : X \to A$ s.t. $f|_A = Id|_A$.

Corollary: There is no retraction from the disk B^2 onto S^1.

Proof: Suppose $\exists$ continuous map $f: B^2 \to S^1$ s.t. $f|_{S^1} = Id|_{S^1}$.



There must be a point left at the end

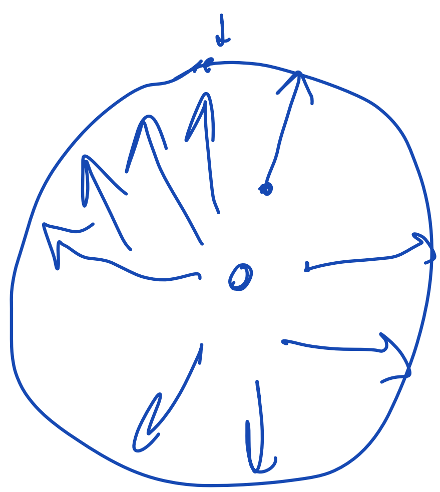
from the assignment, $f_*: \pi_1(B^2, x_0) \to \pi_1(S^1, y_0)$ is surjective.

But $\pi_1(S^1, y_0) = \mathbb{Z}$

and $\pi_1(B^2, x_0) = 0$

How do we "fix" this? Yes.

Consider $B^2 \setminus \{0\}$. Let $f: B^2 \setminus \{0\} \to S^1$ defined by $f(x) = \frac{x}{\|x\|}$ which is a retraction.



$\Rightarrow$ $f_*: \pi_1(B^2 \setminus \{0\}, x_0) \to \pi_1(S^1, y_0)$ is surjective

$\Rightarrow$ $\pi_1(B^2 \setminus \{0\}) \neq 0$

In fact $\pi_1(B^2 \setminus \{0\}, x_0) \cong \pi_1(S^1, x_0) = (\mathbb{Z}, +)$

Post-lecture-Practice-Questions

1) Do the exercises alone.

2) a) Let $f: I \to X$ be a loop based at $x_0 \in X$. Show that $\exists!$ continuous map $\tilde{f}: S^1 \to X$ s.t. $f = \tilde{f} \circ p$ where $p: I \to S^1$
$: x \mapsto (\cos 2\pi x, \sin 2\pi x)$

b) Show that $f \in [e_{x_0}] \iff \tilde{f}$ is nullhomotopic.
Conclude that the identity map from S^1 to S^1 is not nullhomotopic.

3) Solve # 3 in section 54

4) Let $p: E \to B$ be a covering map where E is path-connected.
a) Show that if B is simply connected, then p is a homeomorphism.
b) Suppose

5) Let $p: E \to B$ be a covering map. Suppose $p^{-1}(b)$ is finite $\forall b \in B$.
a) Show that if B is compact then E is compact.
b) Will $|p^{-1}(b)|$ necessarily be the same for each b?
How does $|p^{-1}(b)|$ relate with $|\pi_1(B, b)|$?