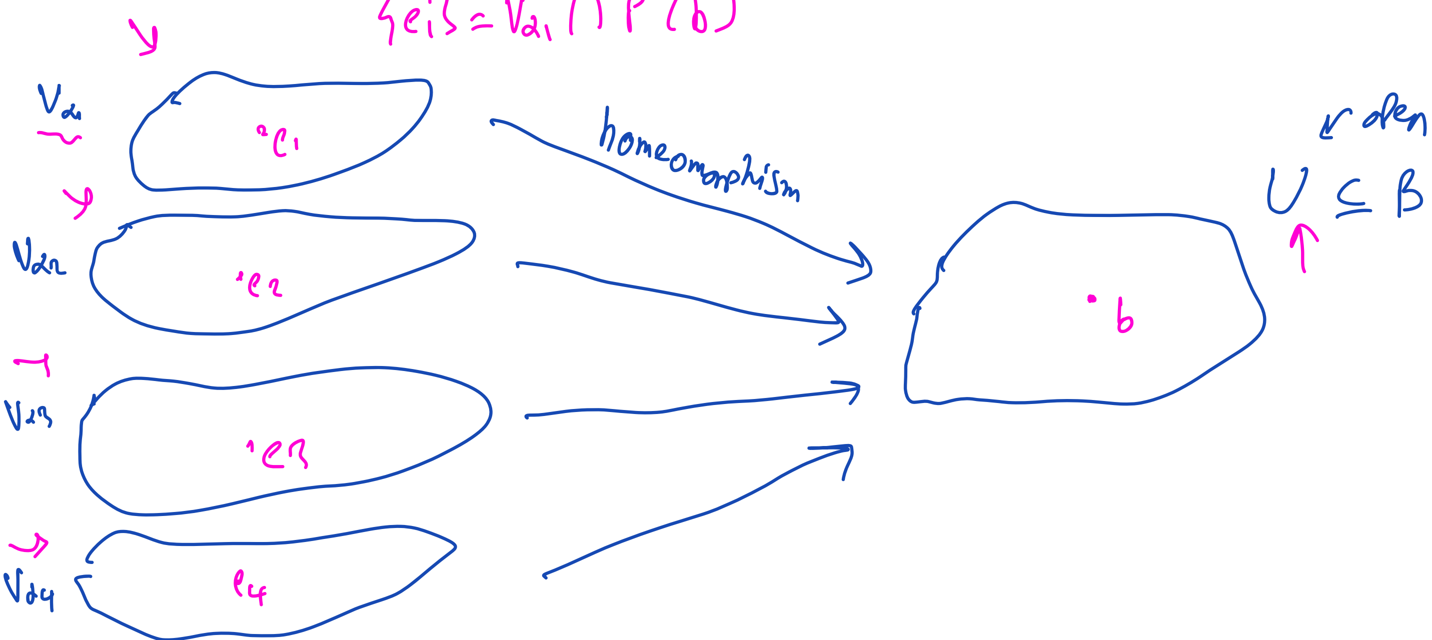
* Course Evaluations
* Ass. 6 is due Sunday Aug 13.

Covering Spaces

Covering spaces is one of the tools that allow us to compute fundamental groups of many topological spaces.

Def: Let $p: E \to B$ be a continuous surjective map. We say that the open set U in B is evenly covered by p if $p^{-1}(U)$ can be written as a disjoint union of open sets V_α in E s.t. for each α, $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Then the collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices.

$\{e_i\} = V_{\alpha_1} \cap p^{-1}(b)$



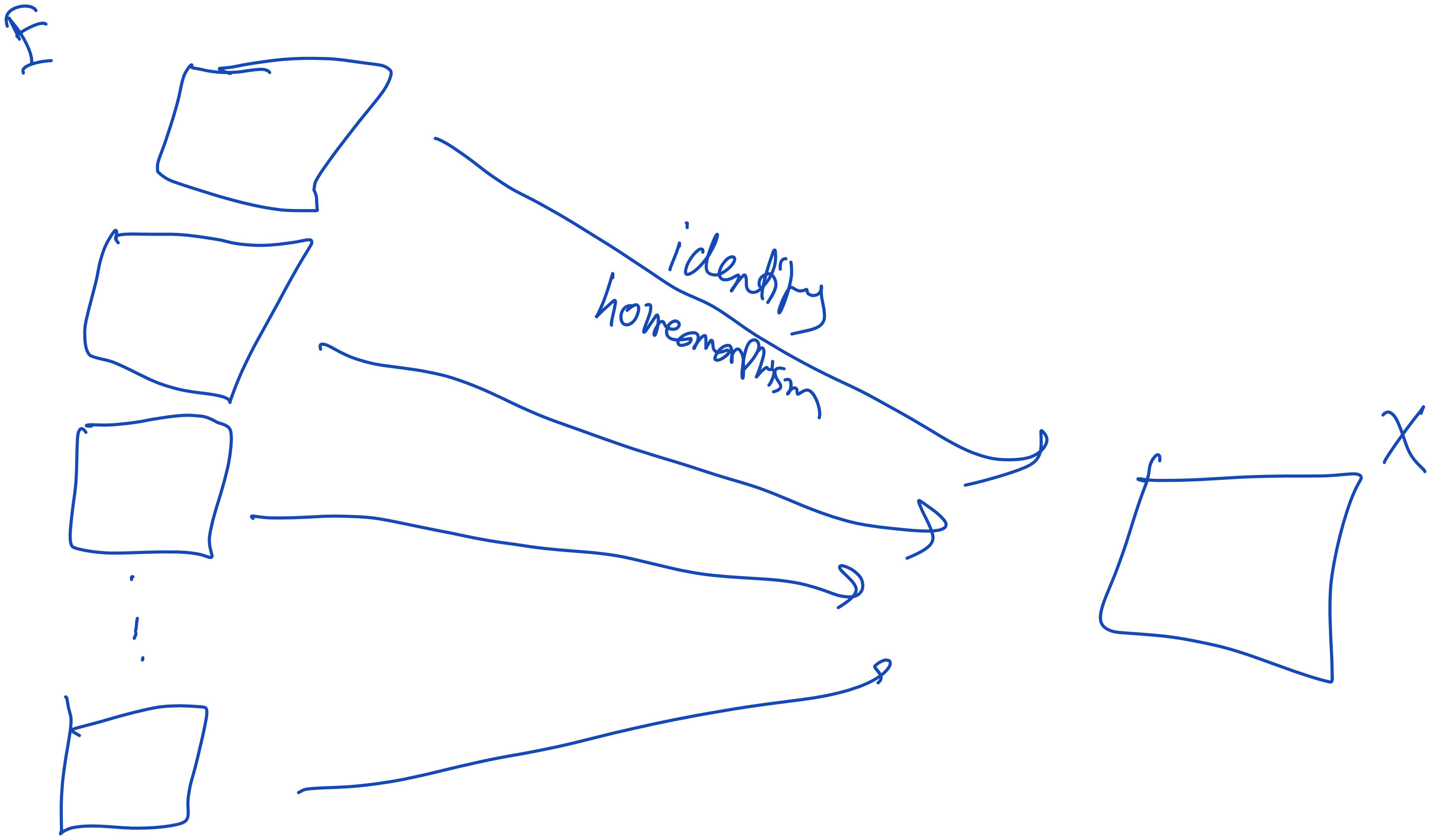
Def: Let $p: E \to B$ a continuous surjective map. We say that p is a covering map if for each $b \in B$, $\exists$ neighbd U of b that is evenly covered by p. We call E a covering space of B.

* The subspace $p^{-1}(b)$ is discrete.
* p is an open map. Let $A \subseteq E$ be open. Let $b_0 \in p(A)$. Then $\exists$ neighbd U of b_0 that is evenly covered by p. Let $e_0 \in A$ s.t. $p(e_0) = b_0$ and let V_α be the slice in the partition of $p^{-1}(U)$ that contains e_0. Then $p|_{V_\alpha}(V_\alpha \cap A)$ is a neighbd of b_0 in $p(A)$ $\Rightarrow$ $p(A)$ is open in B.

* p is a local homeomorphism meaning that $\forall e \in E$, $\exists$ neighbd A of e and neighbd U of $p(e)$ s.t. $p|_A : A \to U$ is a homeomorphism.

Example: Let X be any topological space.
Let $E = X \times \{1, \cdots, n\}$ (n disjoint copies of X)

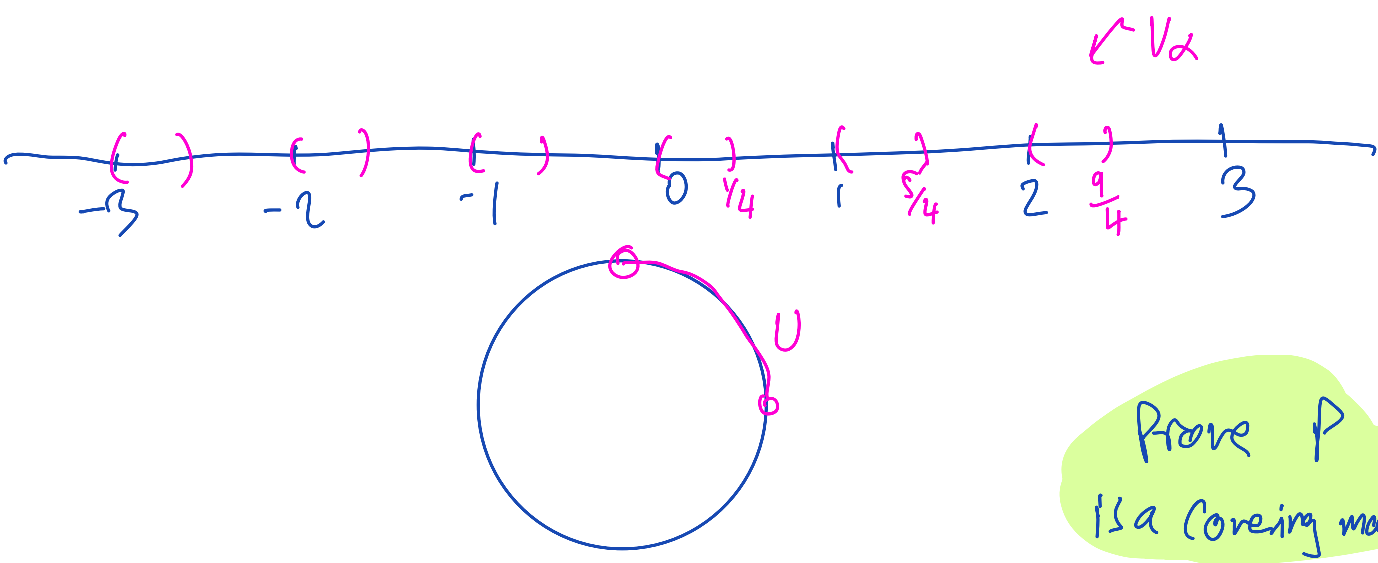
Then $p: E \to X$ defined by $p(x,k) = x$ $\forall (x,k) \in E$ is a (trivial) covering map.



We usually restrict ourselves to path connected covering spaces.

<u>Ex</u> (not trivial):

Define $p : \mathbb{R} \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$



Prove p is a covering map

U is evenly covered by p since $p^{-1}(U) = \bigcup_{n\in\mathbb{Z}} (n, n+\frac{1}{4})$

and $p|_{(n,n+\frac{1}{4})} : (n, n+\frac{1}{4}) \to U$ is a homeomorphism $\forall n \in \mathbb{Z}$.

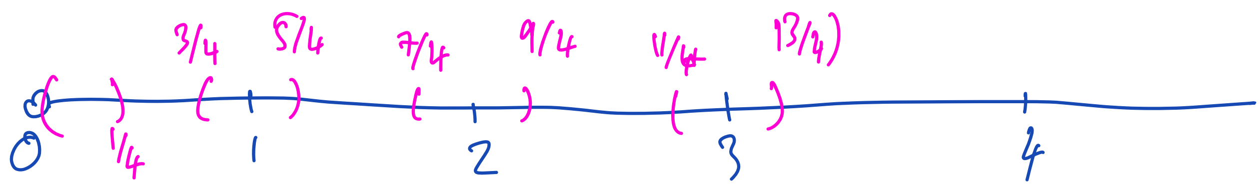
So $\{(n, n+\frac{1}{4})\}_{n\in\mathbb{Z}}$ is a partition of $p^{-1}(U)$ into slices.

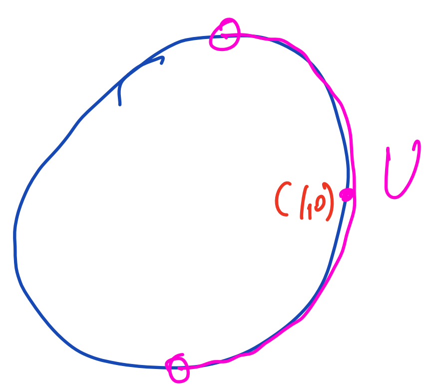
Not every local homeomorphism is a covering map:

<u>Ex</u>: Define $p : (0,\infty) \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$

p is cont, surj and is a local homeomorphism.

But is it a covering map?





Is U evenly covered by p?

$p^{-1}(U) = (0, \frac{1}{4}) \cup (\frac{3}{4}, \frac{5}{4}) \cup (\frac{7}{4}, \frac{9}{4}) \cup \ldots$

Show $\exists$ no neighbd U of $(1,0)$ that is evenly covered.

but $P|_{(0,\frac{1}{4})} : (0,\frac{1}{4}) \to U$ is only an embedding and not a homeomorphism.

Proposition:

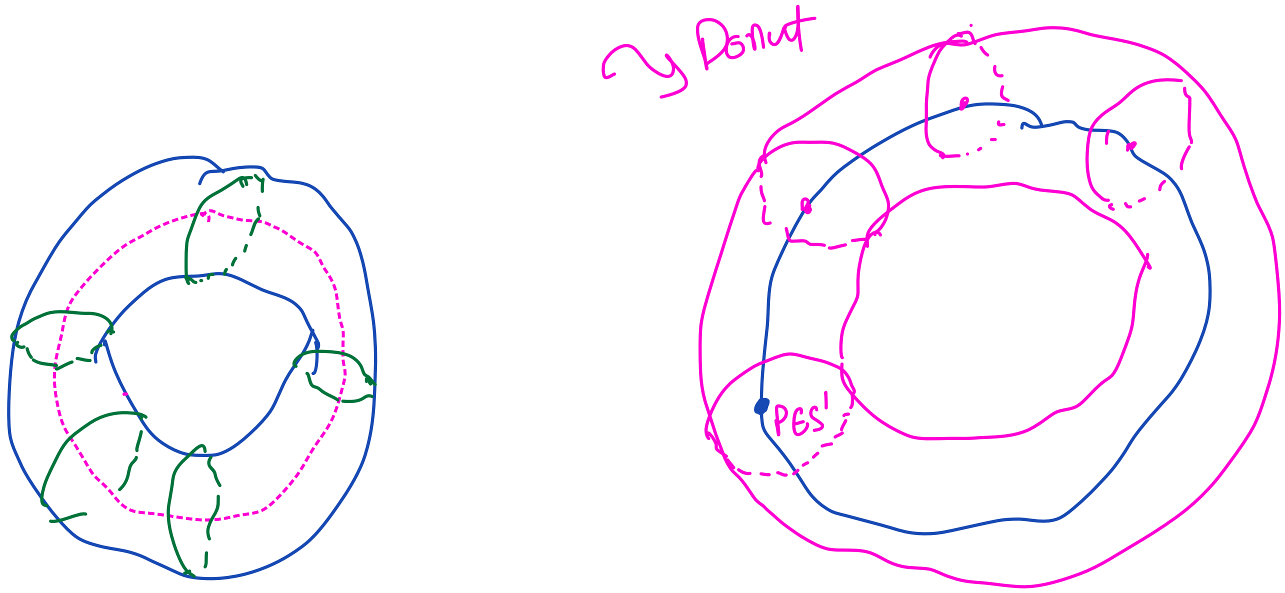
1) Let $P: E \to B$ be a covering map. If B_0 is a subspace of B and $E_0 = P^{-1}(B_0)$, Then $P|_{E_0} : E_0 \to B_0$ is also a covering map.

2) If $P_1 : E_1 \to B_1$ and $P_2 : E_2 \to B_2$ are covering maps, then $P_1 \times P_2 : E_1 \times E_2 \to B_1 \times B_2$ is also a covering map.

Prove this

Ex: Let $P: \mathbb{R} \to S^1$ be the covering map defined above. Then $P \times P : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is also a covering map.

$S^1 \times S^1 \subseteq \mathbb{R}^4$ but admits a nice embedding into $\mathbb{R}^3$.



Think of an embedding of $T := S^1 \times S^1$ into $\mathbb{R}^3$.

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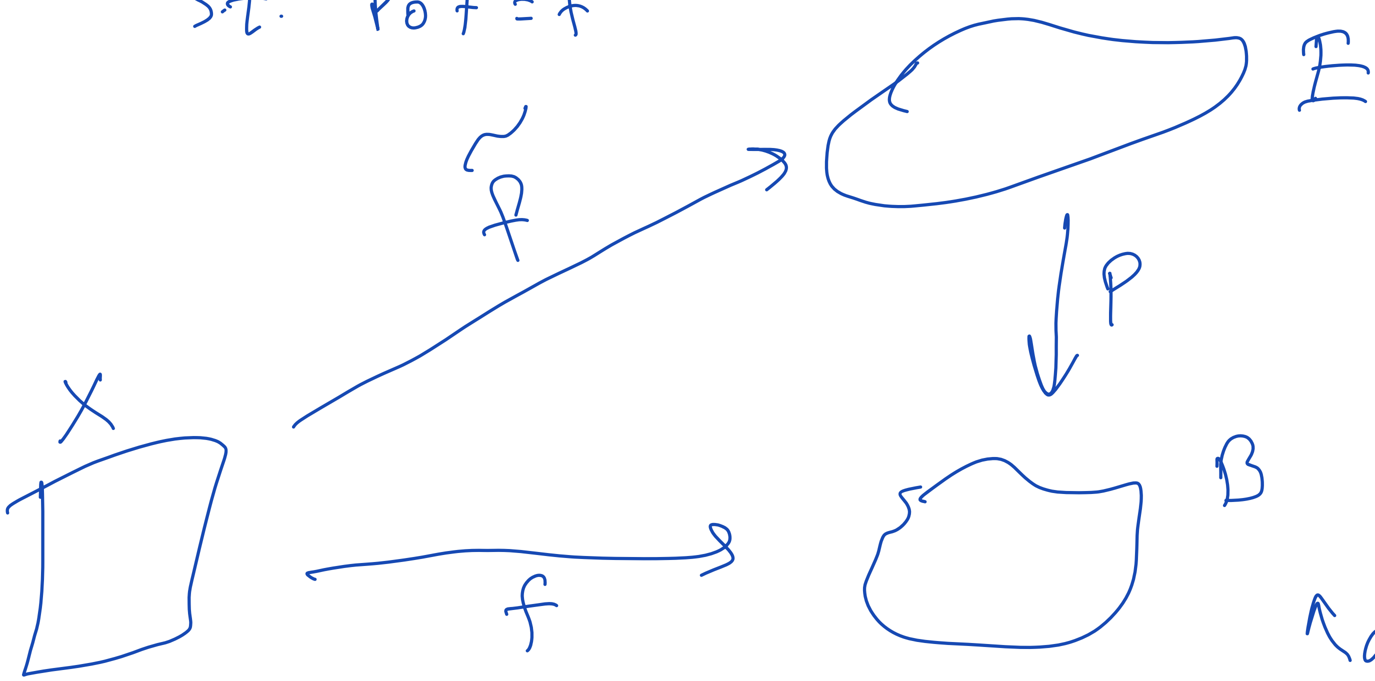
15 min break: fill in Course evaluations.

Def: Let $p: E \to B$ be a map. If $f: X \to B$ is a continuous map, a ==lifting== of f is map $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$



↑ diagram commutes

What makes covering maps a great tool for computing fundamental groups is its ability of lifting path homotopies from B to E.

<u>Lemma</u>: Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$. Any path $f:[0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.

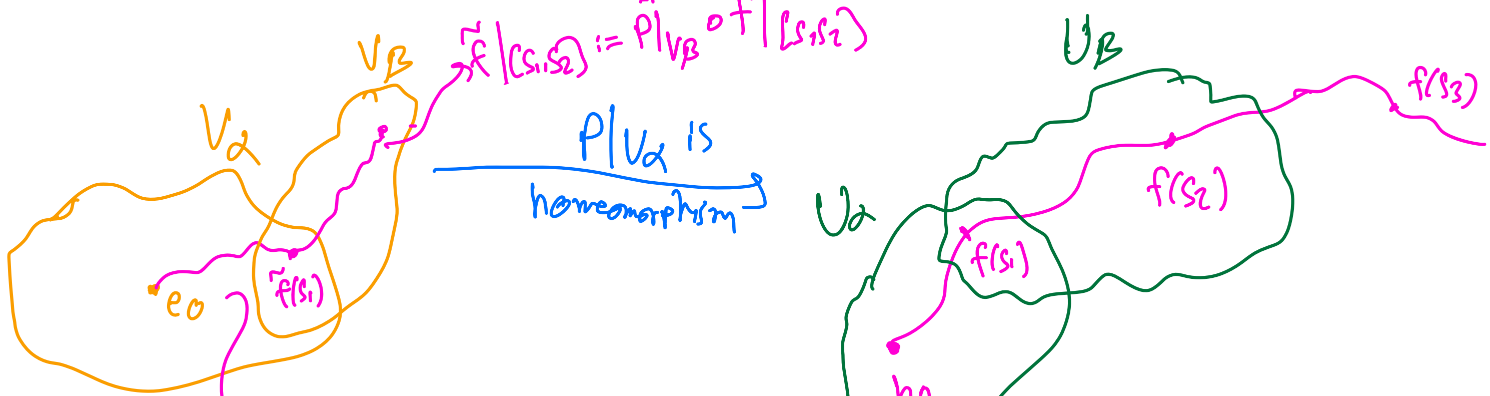
<u>Proof</u>: Cover B with open sets $\{U_\alpha\}_{\alpha \in J}$ that are

evenly covered by p. Can we find a partition of $[0,1]$ into the intervals $[0,s_1], [s_1,s_2], \cdots, [s_{n-1}, s_n]$ s.t. $f|_{[s_i, s_{i+1}]}$ is contained in an open set that is evenly covered? Yes we can.

Lesbegue #

$\{f^{-1}(U_\alpha)\}_{\alpha\in J}$ is an open cover for $[0,1]$ and so by the Lebesgue number lemma, $\exists\, \delta>0$ s.t. for each interval with length $<\delta$, it is contained in $f^{-1}(U_\alpha)$ for some $\alpha\in J$.

Choose $0=s_0 < s_1 < \cdots < s_n = 1$ s.t. $s_{i+1}-s_i < \delta$ and so $f|_{[s_i, s_{i+1}]}$ is contained in U_α for some $\alpha \in J$.



$\tilde f : [0,s_1] \to E$

by $\tilde f|_{[0,s_1]} := p|_{V_\alpha}^{-1} \circ f|_{[0,s_1]}$

Let U_α be neighbd of $f([0,s_1])$ that is evenly covered. Let V_α be the slice in the partition of $p^{-1}(U_\alpha)$ that contains e_0. Then define $\tilde f|_{[0,s_1]} := p|_{V_\alpha}^{-1} \circ f_{[0,s_1]}$.

Then let U_β be neighbd of $f([s_1,s_2])$ that is evenly covered. Let V_β be the slice in the partition of $p^{-1}(U_\beta)$ that contains $\tilde{f}(s_1)$. Define $\tilde{f}|_{[s_1,s_2]} := p|_{V_\beta}^{-1} \circ f|_{[s_1,s_2]}$.

Proceed in this way to define the path $\tilde{f} : [0,1] \to E$.
It clearly satisfies $f = p \circ \tilde{f}$.

The construction of $\tilde{f}$ is completely determined by p and f.

Uniqueness follows from the proof: Let $\tilde{\tilde{f}}$ be another path in E starting at e_0 and satisfying $f = p \circ \tilde{\tilde{f}}$.

Since $\tilde{\tilde{f}}(0) = \tilde{f}(0) = e_0$, then on $[0,s_1]$ $p|_{V_\alpha}$ is a homeomorphism from V_α to U_α. And so $\tilde{\tilde{f}}|_{[0,s_1]} = p|_{V_\alpha}^{-1} \circ f = \tilde{f}|_{[0,s_1]}$.

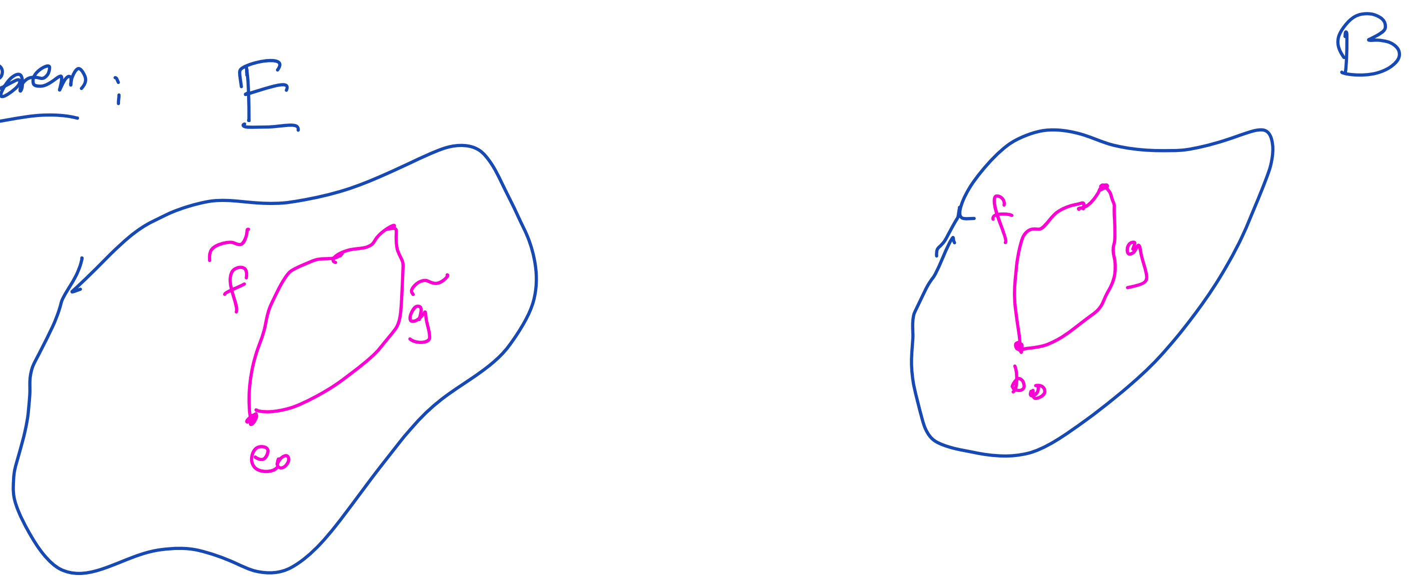
The same holds for any of the other subintervals.

$\boxtimes$

<u>Lemma</u>: Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$.
Let $f : I \times I \to B$ is continuous with $f(0,0) = b_0$.
There is a unique lifting of F to a continuous map $\tilde{F} : I \times I \to E$ s.t. $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is also a path homotopy.

Proof is similar to the previous lemma. Show it rigorously.

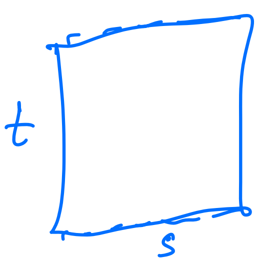
Theorem:



Let $p: E \to B$ be a covering map. Let $p(e_0) = b_0$.
Let $f, g: I \to B$ be paths in B s.t. $f(0) = g(0) = b_0$.
Let $\tilde f, \tilde g: I \to E$ be their respective liftings to paths in E starting at e_0.
If $f \simeq_p g$ then $\tilde f \simeq_p \tilde g$.

Proof: Let $F: I \times I \to B$ be a path homotopy of f and g. Then by the previous lemma $\exists$ a path homotopy $\tilde F: I \times I \to E$ that is a lifting of F s.t. $\tilde F(0,0) = e_0$.

Since $\tilde F$ is a path homotopy,



$\tilde F(0 \times I) = \{e_0\}$ and $F(1 \times I) = \{e_1\}$

Show this: $\tilde{f} := F|_{I\times 0}$ and $\tilde{g} := F|_{I\times 1}$ is the respective lifting of the paths f and g. And so $\tilde{f} \simeq_p \tilde{g}$.

$\blacksquare$

Post-Lecture-Practice-Questions

1) Show the exercises above?

2) In the proof above, why is $\tilde{F}(0\times I) = \{e_1\}$ and $\tilde{F}(1\times I) = \{e_2\}$?
Deduce that $\tilde{f} = \tilde{F}|_{I\times 0}$ and $\tilde{g} = \tilde{F}|_{I\times 1}$

3) Solve #1 & #3 in section 54.

4) Let $p: E \to B$ a covering map where B is locally compact and Hausdorff.
Show that E is also locally compact and Hausdorff.

5) Let $p: E \to B$ be cont & surjective. Suppose U is evenly covered by p. Is the partition of $p^{-1}(U)$ unique?
What if U is connected?