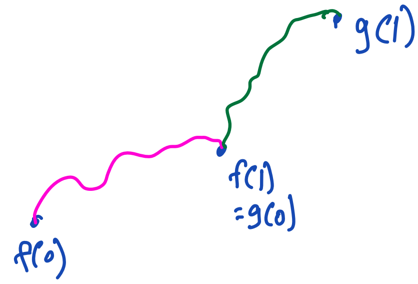
* Course Evaluation

* Required reading: Appendix C from "Topological manifolds"

Fundamental Group

For two paths $f, g: I \to X$ s.t. $f(1) = g(0)$,

we defined $[f] * [g] = [f * g]$



Is {path homotopy classes} a group under the operation $*$?

No!

What if we choose a base point x_0 and consider all paths from x_0 to itself?

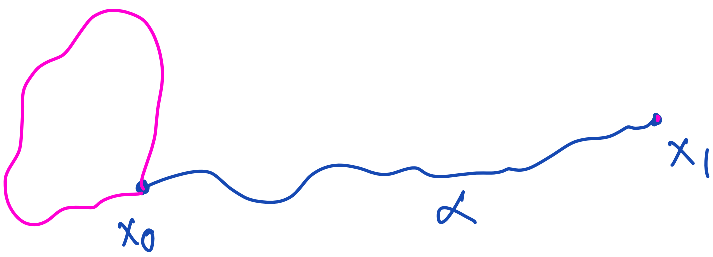
Def: Let $x_0 \in X$. A path from x_0 to itself is called a loop based at x_0. The set of path homotopy classes of loops based at x_0 with the operation $*$ is called the fundamental group of X at the base point x_0.

It is denoted by $\pi_1(X, x_0)$.

Thank to Thm from last lecture, $\pi_1(X, x_0)$ is a group under $*$.

The identity e_{x_0}.

<u>Ex:</u> $\mathbb{R}^2$: Every loop based at x_0 is path homotopic to e_{x_0} $\Rightarrow$ $\pi_1(\mathbb{R}^2, x_0) = \{[e_{x_0}]\}$ is called the trivial group.



recall $\bar{\alpha}(t) := \alpha(1-t)$

<u>Def:</u> Let α be a path from x_0 to x_1. We define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

conjugation

Show $\hat{\alpha}$ is well defined.

<u>Theorem:</u> $\hat{\alpha}$ is a group isomorphism.

<u>Proof:</u> We want to show that $\hat{\alpha}$ is bijective and is a homomorphism ("linear" / perserves the operation)

$*$ $\hat{\alpha}([f] * [g]) = \hat{\alpha}([f * g])$

$= [\bar{\alpha}] * [f * g] * [\alpha]$

$$= [\bar{\alpha} * f * g * \alpha]$$
$$= [\bar{\alpha} * f * e_{x_0} * g * \alpha]$$
$$= [\bar{\alpha} * f * \alpha * \bar{\alpha} * g * \alpha]$$
$$= [\bar{\alpha} * f * \alpha] * [\bar{\alpha} * g * \alpha]$$
$$= \hat{\alpha}([f]) * \hat{\alpha}([g])$$

* $\hat{\alpha}$ is bijective. Consider $\hat{\bar{\alpha}} : \pi_1(X, x_1) \to \pi_1(X, x_0)$

I claim that $\hat{\alpha} \circ \hat{\bar{\alpha}} = Id_{\pi_1(X,x_1)}$

and $\hat{\bar{\alpha}} \circ \hat{\alpha} = Id_{\pi_1(X,x_0)}$ $\leftarrow$

Let $[f] \in \pi_1(X, x_0)$, Then $\hat{\bar{\alpha}} \circ \hat{\alpha}([f])$

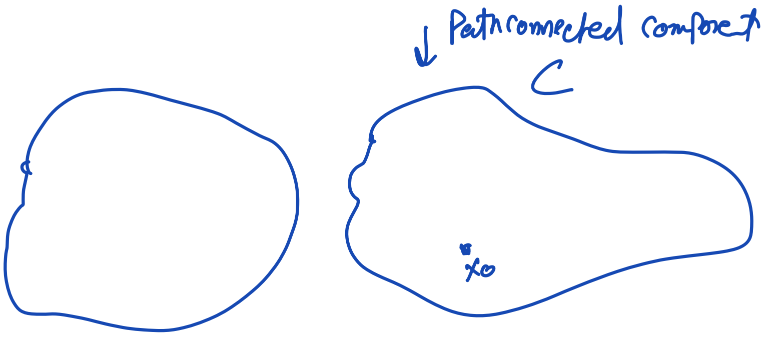
$$= \hat{\bar{\alpha}}([\bar{\alpha} * f * \alpha])$$
$$= [\underbrace{\bar{\bar{\alpha}}}_{\alpha} * \bar{\alpha} * f * \alpha * \bar{\alpha}]$$
$$= [e_{x_0} * f * e_{x_0}]$$
$$= [f]$$

as needed

Similarly, $\forall [f] \in \pi_1(X, x_1)$, $\hat{\alpha} \circ \hat{\bar{\alpha}}([f]) = [f]$

$\Rightarrow$ $\hat{\alpha}$ is bijective. $\blacksquare$

<u>Corollary</u>: If X is path connected and $x_0, x_1 \in X$, Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.



$\pi_1(C, x_0) = \pi_1(X, x_0)$

depends only on the path connected component containing x_0.

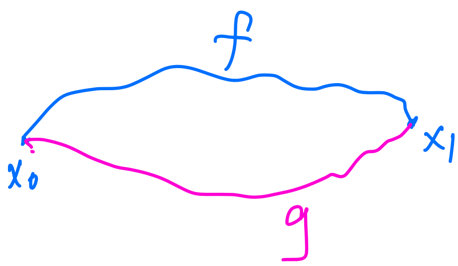
So we will focus on path connected spaces.

Def: A space X is said simply connected if it's path-connected and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$ (and hence for every $x_0 \in X$).

We will denote this by $\pi_1(X, x_0) = 0$.

Proposition: In a simply connected space X, any two paths having the same initial and final point are path homotopic.

Proof



Hint: Then $f * \bar{g} \simeq_p e_{x_0}$

$g * \bar{f} \simeq_p e_{x_1}$

$$[f] = [f] * [e_{x_1}] = [f] * [\bar{g} * g] = [f * \bar{g}] * [g] = [e_{x_0}] * [g] = [g]$$

$$\therefore [f] = [g].$$

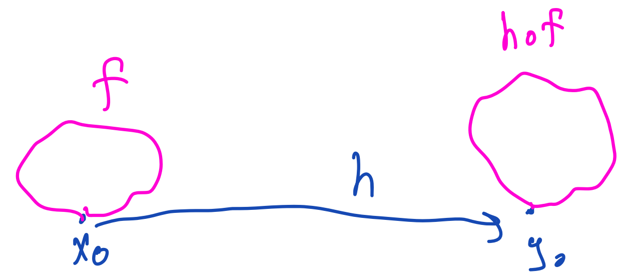
Let $h : X \to Y$ s.t. $h(x_0) = y_0$.

We denote this by $h : (X, x_0) \to (Y, y_0)$

Def: Let $h: (X, x_0) \to (Y, y_0)$ be a continuous map.

Define $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by

$h_*([f]) = [h \circ f]$



Show this is well defined.

Also $h_*([f] * [g])$

$= [h \circ f * g]$

$= [h \circ f * h \circ g]$

$= [h \circ f] * [h \circ g]$

$= h_*([f]) * h_*([g])$

and so h_* is a homomorphism.

Each continuous map $h: (X, x_0) \to (Y, y_0)$ induces a group homomorphism $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

Proposition:

* Let $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ be cont maps. Then $(k \circ h)_* = k_* \circ h_*$.
* Let $Id: (X, x_0) \to (X, x_0)$ be the identity map. Then $Id_*: \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity map on $\pi_1(X, x_0)$
* Let $h: (X, x_0) \to (Y, y_0)$ be a homeomorphism. Then h_* is a group isomorphism

Prove this

Post-Lecture-Practice-Questions.

1) Do the exercises above

2) Show that $[\bar{\alpha}] = [\alpha]^{-1}$ $\forall \alpha \in \pi_1(X, x_0)$.
Interpret the fact $\bar{\alpha} * \alpha = e_{x_0}$ visually.

3) Solve #1 - #5 in Munkres Section 52.