Algebraic Topology

Given two topological space, we wish to know whether or not they're homeomorphic.

Studying the topological properties / topological invariants is a way to decide if two spaces are homeomorphic.

compact $\leftarrow$ $S^1 \not\cong \mathbb{R}$ $\leftarrow$ not compact

$$\begin{array}{ccc} S^1 & \not\cong & \mathbb{R} \\ \downarrow & & \downarrow \\ [0,1] & \not\cong & S^1 \end{array}$$

(studying the cut points of each space)

$\mathbb{R} \not\cong \mathbb{R}^2$ (cut points)

The topological properties we have studied so far do not help us

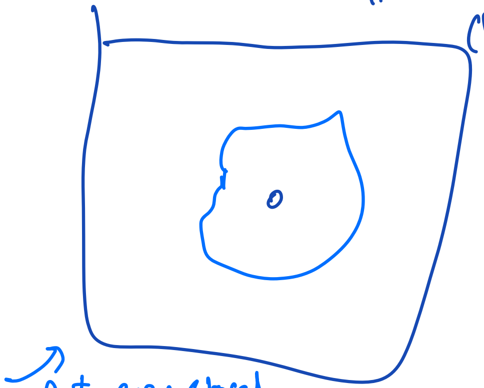
$\rightarrow$ $\mathbb{R}^2$ $\mathbb{R}^2 \setminus \text{point}$

$\rightarrow$ $\mathbb{R}^2$ $\mathbb{R}^3$

$\rightarrow$ S^2 $T^2 = S^1 \times S^1$

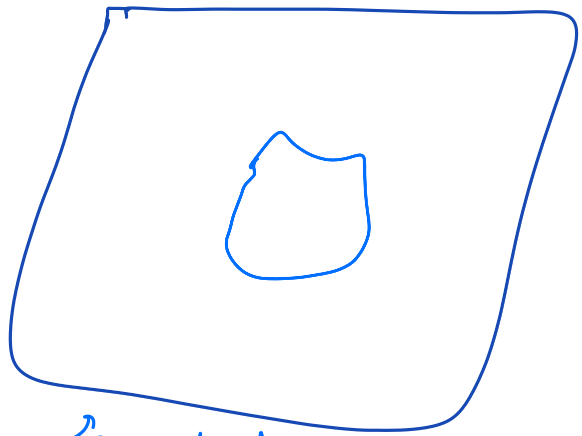
So we need new tools & techniques.

$\mathbb{R}^2 \setminus$ point (Punctured plane)



not every closed curve can be cont. deformed into a point

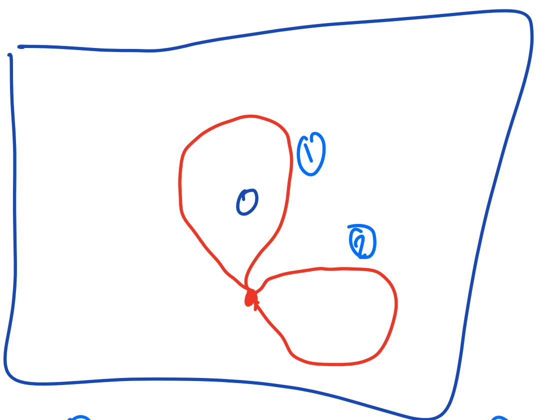
$\mathbb{R}^2$



every closed curve can be cont. deformed into a point.

A space is simply connected if every closed curve can be deformed into a point.

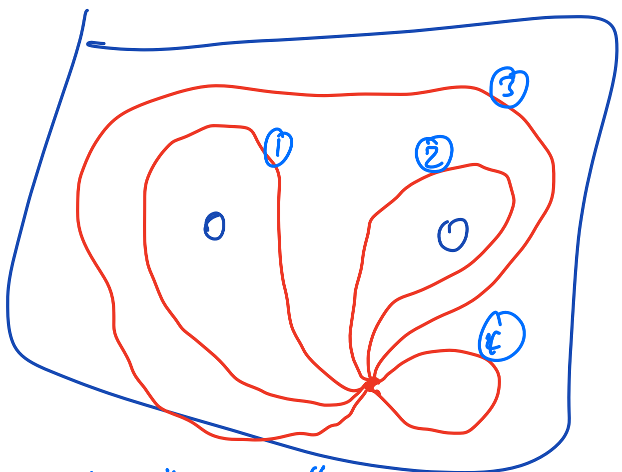
$\mathbb{R}^2 \setminus$ Point



1 cannot be deformed into 2

2 "different" curves

$\mathbb{R}^2 \setminus$ 2 Points



4 "different" curves

Studying all the "different" closed curves (loops) will introduce a new topological invariant that is more general than simple connectedness.

2 Curves are the "same" if one can be cont. deformed into the other. This will define an equivalence relation on the space of curves/paths. We will define an operation on {equivalence classes} giving it an algebraic structure (making it a group called the fundamental group). It will turn out that homeomorphic spaces have the "same" fundamental group.
This introduces a new tool to prove 2 spaces aren't homeomorphic (by studying their fundamental group).

In short, Algebraic topology is the study of topological spaces by means of algebraic objects.

Homotopy of Paths

__Def:__ If $f, f' : X \to Y$ are continuous functions, we say f is homotopic to f' if $\exists$ cont map $F : X \times I \to Y$ s.t. ($I = [0,1]$)

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = f'(x)$$

for each $x \in X$.

The map F is called a homotopy between f and f'.

If f is homotopic to f', we write $f \simeq f'$.
If $f \simeq f'$ and f' is a constant map, we say f is **nulhomotopic**.

F is a cont 1-Parameter family of maps from X to Y.

$$t \longmapsto F_t \left(: x \longmapsto F(x,t) \right)$$

$$F_0 = f \quad , F_1 = f'$$

Def A continuous $f: I \to X$ is called a **Path** in X.
$x_0 := f(0)$ is called the **initial point**.
$x_1 := f(1)$ is called the **final point**.

Def: Two paths f and f' are **Path homotopic** if they have the same initial and final point and there exist a cont map $F: I \times I \longrightarrow X$ s.t.

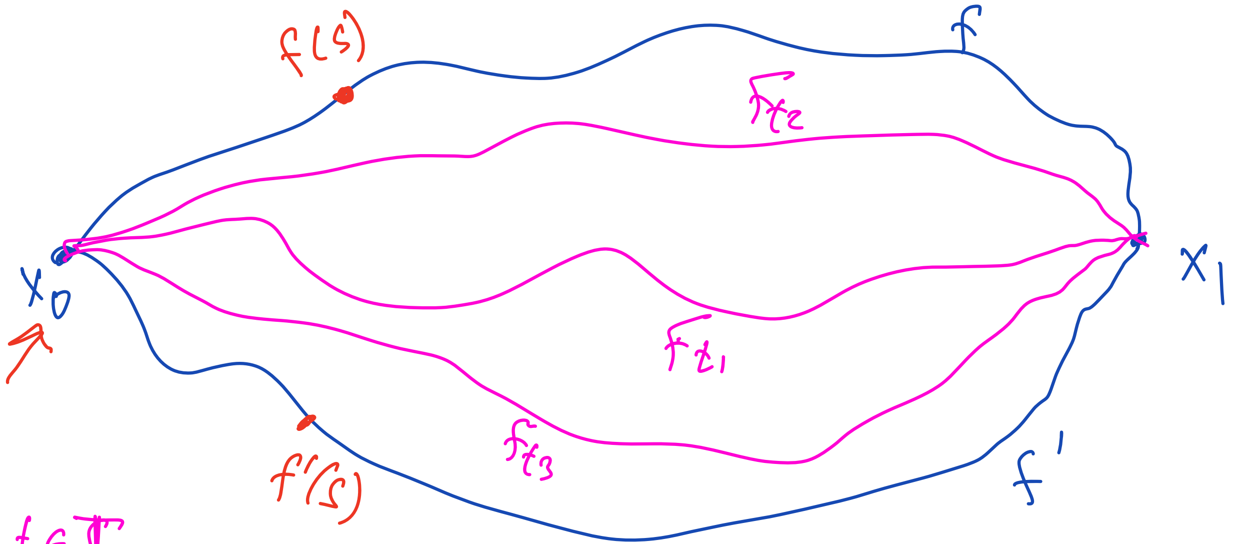
time parameter

$$F(s,0) = f(s) \quad , \quad F(s,1) = f'(s)$$

$$F(0,t) = x_0 \quad , \quad F(1,t) = x_1$$

time tth path time tth paths

$\forall\, s,t \in I$.



for every $t \in I$,

$F_t : s \longmapsto F(s,t)$

So $t \longmapsto F_t\,(s \mapsto F(s,t))$ is a cont 1-parameter family of paths all starting from x_0 and ending at x_1.

We call F a path homotopy between f and f' and we write $f \simeq_p f'$.

<u>Lemma</u>: $\simeq$ and $\simeq_p$ are equivalence relations.

Proof

Reflexive: Define $F : X \times I \to Y$
$F(x,t) = f(x)$

Symmetric: suppose $f \simeq f'$ so $\exists$ homotopy F from f to f'. Define $G : X \times I \to Y$

by $G(x,t) = F(x, 1-t)$

transitive: Suppose $f \simeq f'$ with homotopy F
and $f' \simeq f''$ with homotopy G

Then define $H: X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t), & 0 \le t \le \frac{1}{2} \\ G(x, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Then H is the desired homotopy.

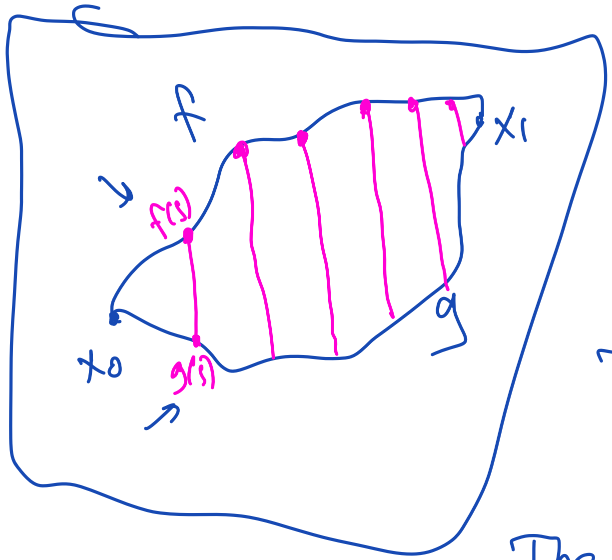
(show H is cont)

Examples: * let $f, g : X \to \mathbb{R}^2$ be cont.

Then $F(x,t) = (1-t) f(x) + t\, g(x)$ is a homotopy from f to g.

Straight-line homotopy

* let f, g be paths in $\mathbb{R}^2$ starting from x_0 and ending at x_1.



Define $F: I\times I \to \mathbb{R}^2$ by

$F(s,t) = (1-t)f(s) + t g(s)$

Then F is a homotopy from f to g.

Then any two paths with the same initial and final point are homotopic.

More generally, any convex subset of $\mathbb{R}^n$ will satisfy.

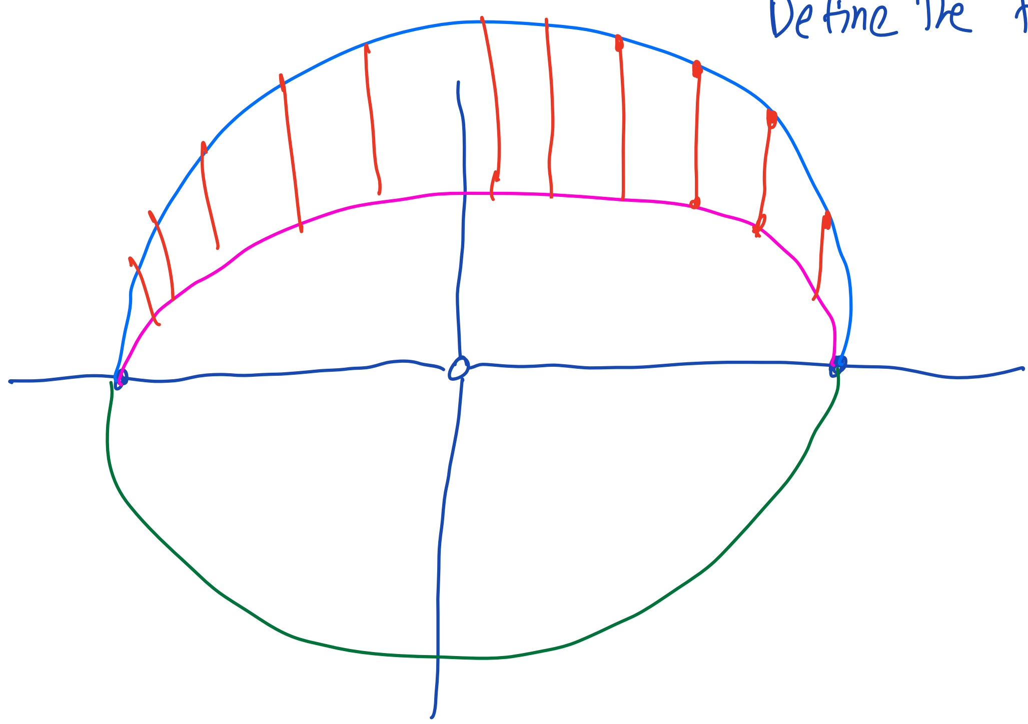
$\times$ Consider the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$

Define the following three paths:

$f(s) := (\cos \pi s, \sin \pi s)$

$g(s) := (\cos \pi s, 2\sin \pi s)$

$h(s) := (\cos \pi s, -\sin \pi s)$



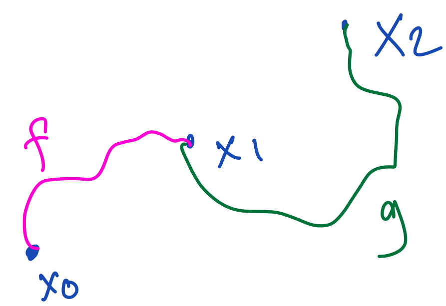
Observe $F(s,t) = (1-t)f(s) + t g(s)$ is a path homotopy from f to g, so $f \simeq_p g$.

Is $f \simeq_p h$? Intuitively no. (Proven later).

We now introduce an algebraic operation.

Def: If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2, then we define the product $f * g$ to be the path h defined by:

$$h(s) := \begin{cases} f(2s), & 0 \le s \le 1/2 \\ g(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$



show that h is cont. and so is a path from x_0 to x_2.

for a path f in X, let $[f]$ be the equivalence class containing f wrt $\simeq_p$. called path homotopy class.

The product operation induces a well defined operation on the path homotopy classes:

$$[f] * [g] = [f * g]$$

To verify that $*$ is well defined, we need to show that if $f' \simeq_p f$ and $g' \simeq_p g$, then $f' * g' \simeq_p f * g$.

Show this.

<u>Theorem</u>: The operation $*$ has the following properties:

1) (Associativity) $\big([f] * [g]\big) * [h] = [f] * \big([g] * [h]\big)$

whenever the above is well defined.

2) (Right and left identity). Given $x \in X$, let e_x be the constant path $e_x : I \to X$ defined by $e_x(s) = x \quad \forall s \in I$.

If f is a path from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f]$$

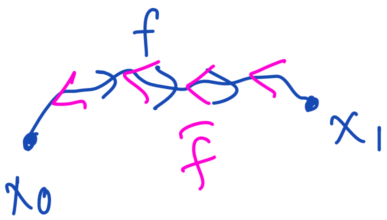
$(f * e_{x_1} \simeq_p f$ and $e_{x_0} * f \simeq_p f)$

3) (Inverse) Given a path f from x_0 to x_1,

define $\bar{f} : I \to X$ by $\bar{f}(s) = f(1-s)$ to be the reverse of f.

Then $[f] * [\bar{f}] = [e_{x_0}]$

$[\bar{f}] * [f] = [e_{x_1}]$



Prove

Intro to Groups

A group is a set G together with an operation

$\cdot : G \times G \longrightarrow G$ satisfying:

1) Associative. $(a \cdot b) \cdot c = a \cdot (b \cdot c) \; (= a \cdot b \cdot c)$

2) $\exists$ an element $e \in G$ that satisfies:

$$g \cdot e = e \cdot g = g \quad \forall g \in G.$$

e is called the identity.

3) $\forall g \in G$, $\exists h \in G$ s.t. $g \cdot h = h \cdot g = e$

h is denoted by g^{-1} and is called the inverse of g.

Ex: $(\mathbb{R}, +)$ is a group. $(e = 0, \; g^{-1} = -g)$

$(\mathbb{R} \setminus \{0\}, \cdot)$ $\quad (e = 1, \; g^{-1} = \frac{1}{g})$

$(\mathbb{Z}, +)$ is a group

$(\mathbb{N}, +)$ is not a group.

Def: H is a subgroup of G if $H \subseteq G$ and H is a group under the same operation defined on G.

Def: Let G_1 and G_2 be groups and $\phi : G_1 \to G_2$ be a map. We say ϕ is a homomorphism (linear) if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad \forall g_1, g_2 \in G$.

Def: G_1 is isomorphic to G_2 if $\exists$ bijective homomorphism ϕ from G_1 to G_2.

Show that ϕ^{-1} is also a homomorphism.

We say ϕ is an Isomorphism.

Def: Let $\phi : G_1 \to G_2$ be a homomorphism.
The kernel of ϕ is defined by $\ker \phi := \{g \in G_1 \mid \phi(g) = e_2\}$

Show $\ker \phi \leq G_1$

Post-Lecture-Practice-Questions

1) Do the exercises above
2) Prove the pasting lemma: Let $A, B \subseteq X$ be closed sets and let $g_1 : A \to Y$ and $g_2 : B \to Y$ be continuous functions s.t. $g_1|_{A \cap B} = g_2|_{A \cap B}$.

Show that the function $f: A \cup B \to Y$ defined by
$f(x) := \begin{cases} g_1(x), & x \in A \\ g_2(x), & x \in B \end{cases}$ is a well defined continuous function.

3) We will prove the theorem done in lecture regarding the properties of $*$.

a) Let $i : I \to I$ be the identity function, which is also a path from 0 to 1 in I. Show that any other path in I from 0 to 1 is path homotopic to i.

Hint: I is convex.

b) Let f be a path in X from x_0 to x_1. Use (a) to show that $f \simeq_p e_{x_0} * f \simeq_p f * e_{x_1}$. Conclude statement (2) in the theorem.

c) Show that any path from 0 to 0 in I is path homotopic to the constant path e_0.

Use this to show that $f * \bar{f} \simeq_p e_{x_0}$ and $\bar{f} * f \simeq_p e_{x_1}$

Conclude statement (3) in the theorem.

d) Let f be a path from x_0 to x_1. A reparametrization of the path f is another path $f \circ \varphi$ where $\varphi : I \to I$ is a path from 0 to 1.

Show that $f \simeq f \circ \varphi$.

e) Let f, g, h be paths in X s.t. $f(1) = g(0)$ and $g(1) = h(0)$.

Show that $(f * g) * h$ and $f * (g * h)$ are reparametrization of each other and hence path homotopic. Conclude statement (3) in thm.

4) Solve #3 in section 5.1