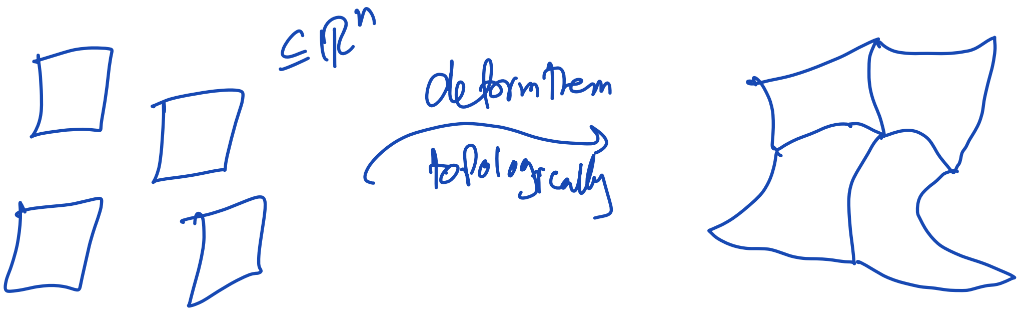
* PS4 Sol are on course website.

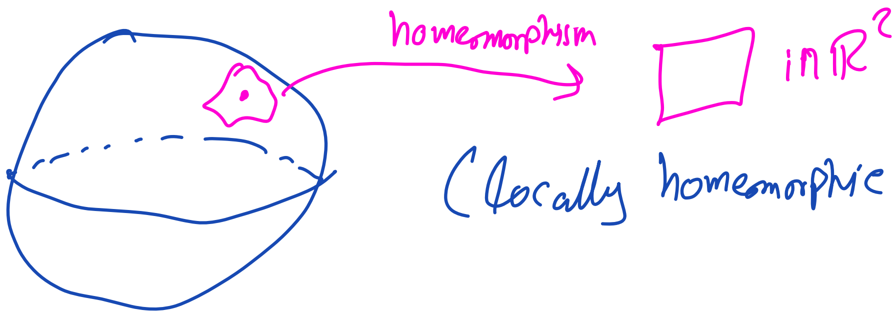
* Exam: Aug 23 9AM → 12PM

Topological Manifold

Manifolds are generalization of Euclidean spaces. They are topolog. spaces that created by sewing deformed patches of open subsets of $\mathbb{R}^n$.

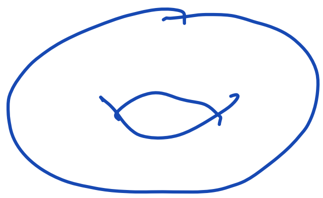


$S^2 \ncong \mathbb{R}^2$

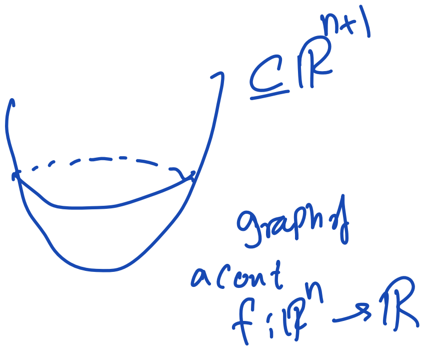


(locally homeomorphic to $\mathbb{R}^2$)

$T^2 = S^1 \times S^1$



locally homeomorphic to $\mathbb{R}^2$



example of a space that is locally homeomorphic to $\mathbb{R}^n$

Möbius strip, Klein bottle, $\mathbb{R}P^n$

<u>Def</u>: We say a topological space X is locally Euclidean of dimension n if $\forall x \in X$, $\exists$ a neighbd U of x and an open subset $V \subseteq \mathbb{R}^n$ and a homeomorphism $\phi: U \to V$.

Any such space will share with $\mathbb{R}^n$ all the local topological properties (locally compact / connected) but not necessarily global
first countability

Properties (compactness, separation axioms, second countable, metrizable)

<u>Def</u>: An n-manifold X is a Hausdorff second countable space that is locally Euclidean of dimension n.

A 1-manifold is called a curve
A 2-manifold is called a surface

<u>Proposition</u>: Every manifold is metrizable.

<u>Proof</u>: Hint: show X is regular.

It turns out that every n-manifold can be embedded in $\mathbb{R}^{2n}$!!
(sharp result)

Def: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open cover for a topological space X. A partition of unity subordinate to $\mathcal{U}$ is a family of continuous functions $\psi_\alpha : X \to [0,1]$ indexed by J and satisfies:

i) $\operatorname{supp} \psi_\alpha \subseteq U_\alpha \quad \forall \alpha \in J$, where $\operatorname{supp} \psi_\alpha = \overline{\{x \in X \mid \psi_\alpha(x) \neq 0\}}$

ii) $\forall x \in X$, $\exists$ neighbd of x that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many $\alpha \in J$. (so $\psi_\alpha(x) = 0$ for all but finitely many $\alpha \in J$)

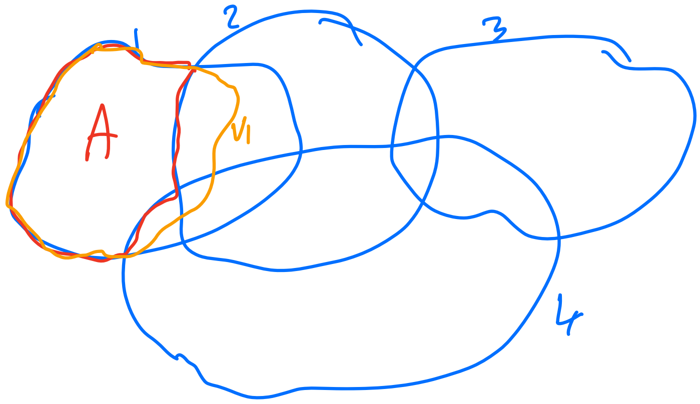
iii) $\displaystyle\sum_{\alpha \in J} \psi_\alpha(x) = 1 \quad \forall x \in X.$

(This is a powerful used to construct global objects from many locally defined objects).

Theorem: Let X be a compact n-manifold and let $\{U_i\}_{i=1}^m$ be an open cover for X. $\exists$ partition of unity subordinate to that open cover.

proof: X is normal since it's compact and Hausdorff.

We will shrink $\{U_i\}_{i=1}^m$ to another open cover $\{V_i\}_{i=1}^m$ that satisfies $\overline{V_i} \subseteq U_i \quad \forall i = 1, \cdots, m.$



$A = X \setminus \left(\bigcup_{i=2}^{n} U_i \right)$ which is closed.

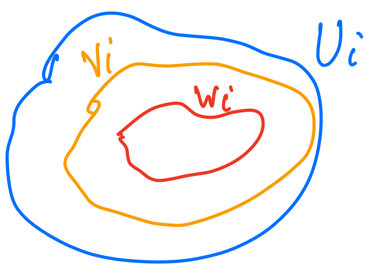
$A \subseteq U_1$ and so $\exists$ a neighbd V_1 of A s.t. $\overline{V_1} \subset U_1$

Then the collection $\{V_1, U_2, \cdots, U_m\}$ is another open cover. We proceed by induction: given open sets $V_1, \cdots, V_{k-1}$ s.t. $\{V_1, \cdots, V_{k-1}, U_k, U_{k+1}, \cdots, U_m\}$ is an open cover, we define $A = X \setminus \left(\bigcup_{i=1}^{k-1} V_i \cup \bigcup_{i=k}^{m} U_i \right)$.

Then A is a closed and U_k is a neighbd of A. Then $\exists$ neighbd V_k of A s.t. $\overline{V_k} \subseteq U_k$.

So we shrunk $\{U_i\}_{i=1}^{m}$ to $\{V_i\}_{i=1}^{m}$ s.t. $\overline{V_i} \subseteq U_i$.

Repeat this procedure to define another open cover $\{W_i\}_{i=1}^{m}$ s.t. $\overline{W_i} \subseteq V_i$



Then using Urysohn lemma, choose for each $i = 1, \cdots, m$ a cont function $\psi_i : X \to [0,1]$ satisfying $\psi_i(\overline{W_i}) = \{1\}$ and $\psi_i(V_i^c) = \{0\}$.

Define $\phi_i(x) := \dfrac{\psi_i(x)}{\sum \psi_i(x)}$. Show that $\{\phi_i\}_{i=1}^{m}$ is the desired

$i=1$ $\to 0$

Partition of unity.

Thm: Embedding Thm:

Every compact n-manifold can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Post-Lecture-Practice-Questions

1) Do the exercises above.
2) We will prove the embedding Thm. Let X be an m-dim compact manifold.

   a) Show that $\exists$ an open cover $\{U_i\}_{i=1}^n$ and embeddings $g_i : U_i \to \mathbb{R}^m$ for $i=1,\dots,n$

   b) Let $\{\phi_i\}_{i=1}^n$ be a partition of unity subordinate to the open cover $\{U_i\}_{i=1}^n$.

   For $i=1,\dots,n$, define $h_i : X \to \mathbb{R}$ by

   $$h_i(x) = \begin{cases} \phi_i(x)\, g_i(x), & x \in U_i \\ 0, & x \notin U_i \end{cases}$$

   Show that h_i is continuous.

   c) Define $F : X \to \mathbb{R}^n \times (\mathbb{R}^m)^n$ by

   $$F(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$$

Show that F is an embedding.

3) Solve #5 in Section 36.

4) a) Let $A \subseteq \mathbb{R}^n$ be closed. Find a cont. function $f: \mathbb{R}^n \to [0,\infty)$ s.t. $f^{-1}(0) = A$.

b) Let X be a compact n-manifold. Let $A \subseteq M$ be a closed set. Show that $\exists$ continuous function $f: X \to [0,\infty)$ s.t. $f^{-1}(0) = A$.