* OH & Tutorials next week.

* Post lecture Practice questions

exercises in lectures

Let X be an arbitrary set

Def: A topology on X is a collection $\mathcal{T} \subseteq P(X)$ satisfying

(1) $\phi, X \in \mathcal{T}$

(2) Unions of sets in $\mathcal{T}$ are also in $\mathcal{T}$ (closed under unions)

(3) Finite intersections of sets in $\mathcal{T}$ are also in $\mathcal{T}$ (closed under finite intersection)

These sets are called open sets.

The pair $(X, \mathcal{T})$ is called topological space.

A topological Property is one only that only depends on the choice of open sets (only depends on $\mathcal{T}$)

Example of topological Prop:

* open / closed (open/closed maps)
* Continuous functions
* limit Points, boundary Points, interior Points, Isolated Points

We will study each very rigorously

* ~~boundedness~~ (requires a notion of distance between points called a metric)
* Compactness
* Connectedness (Path connectedness)
* Convergence of sequence
* ~~Smooth~~ (requires a differential structure)
* ~~Completeness~~ (requires a metric)
* ~~Uniform continuity~~ (")

Examples of Topologies:

* Discrete Topology $\mathcal{T}_{discrete} = P(X)$
* Indiscrete Topology $\mathcal{T}_{indiscrete} = \{\phi, X\}$
* Co-finite Topology $\mathcal{T}_{co\text{-}finite} = \{U \subseteq X \mid U^c \text{ is finite}\} \cup \{\phi\}$
* Co-Countable Topology $\mathcal{T}_{co\text{-}countable} = \{U \subseteq X \mid U^c \text{ is countable}\} \cup \{\phi\}$

Exc: show they are indeed topologies

* Standard topology on $\mathbb{R}^n$ $\mathcal{T}_{standard} = \{\text{unions of open balls}\}$

Show $\{\text{open balls}\}$ are not a topology (open ball are defined using the metric $d(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$

Comparing Topologies:

Def: Let τ_1 and τ_2 be two topologies X.
We say τ_1 is finer than τ_2 if $\tau_1 \supseteq \tau_2$
We say τ_1 is coarser than τ_2 if $\tau_1 \subseteq \tau_2$
We say τ_1 and τ_2 are comparable if $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$.

Basis of a Topology

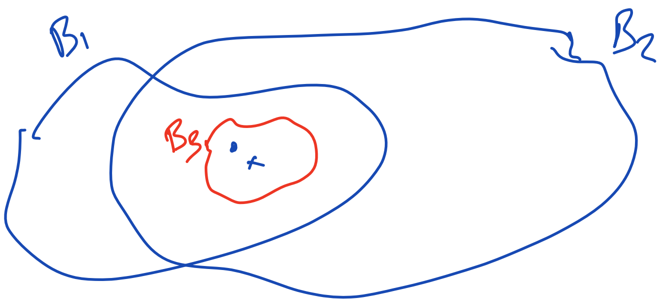
It is easier to restrict ourselves to a smaller collection of open sets that somehow "generate" the topology.

Def: Let X be a set. A basis for a topology on X is a collection $\mathcal{B} \subseteq P(X)$ (called basis sets) satisfying:

(1) $\bigcup_{B\in\mathcal{B}} B = X$

(2) If $x \in B_1 \cap B_2$, Then $\exists\, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

basis sets



The topology generated by $\mathcal{B}$ is defined as

$$\tau_{\mathcal{B}} = \{ U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U \}$$

show this: $$= \{ U \subseteq X \mid U \text{ is a union of sets in } \mathcal{B} \} \cup \{ \emptyset \}$$

Notice that $\mathcal{B} \subset \tau_{\mathcal{B}}$ & so basis sets are also open themselves.

We need to show that $\tau_{\mathcal{B}}$ is a topology:

(1) trivially. $\emptyset, X \in \tau_{\mathcal{B}}$

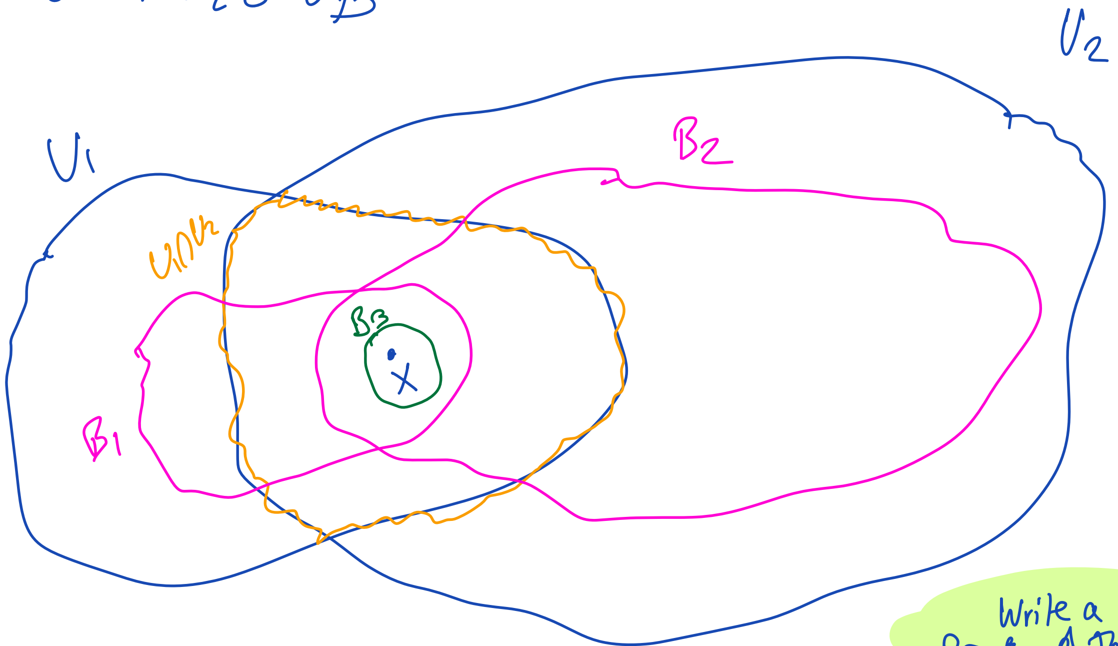
(2) trivially closed under unions

(3) It suffices to show that $\forall U_1, U_2 \in \tau_{\mathcal{B}}$, $U_1 \cap U_2 \in \tau_{\mathcal{B}}$.

(why?)

(use induction to justify)

Let $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$



Write a rigorous proof of this

Ex: $\mathcal{B} = \{ B_r(x) \mid x \in \mathbb{R}^n, r > 0 \}$ is a basis for the standard topology on $\mathbb{R}^n$.

What if we start with a topology? How do we find a basis for it or how do we know a given basis generates that topology.

Lemma: Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B} \subseteq \mathcal{T}$ is a basis that generates $\mathcal{T}$ iff every $U \in \mathcal{T}$ is a union of elements in $\mathcal{B}$.

Proof: ($\Longrightarrow$) by def

($\Longleftarrow$)

Post lecture Practice Questions

1) Do the exercises above.

2) Are $\mathcal{T}_{\text{co-finite}}$ and $\mathcal{T}_{\text{co-countable}}$ comparable? which is finer? When are they equal? When are each equal to $\mathcal{T}_{\text{discrete}}$?

3) Consider this other metric on $\mathbb{R}^n$ defined by $d(x,y) := \max_{1 \le i \le n} |x_i - y_i|$. Consider the topology generated by open balls wrt this metric. Show that this topology is the standard topology on $\mathbb{R}^n$.

4) Define $\mathcal{B} = \{ [a,b) \mid a,b \in \mathbb{R} \}$ show that this is a basis for a topology on $\mathbb{R}$. This topology is called the lower limit topology. How is this comparable to the standard topology on $\mathbb{R}$?