* Assignment 5.

Urysohn lemma: Let A and B be disjoint closed sets in a normal topological space. $\exists$ a cont function $f: X \to [a,b]$ s.t. $f(A) = \{a\}$ and $f(B) = \{b\}$

Remark: Let (X, d) be a metric space. Then for disjoint closed sets A and B, we can define

$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}.$$

Show f satisfies the conclusion of Urysohn lemma.

Def: Let X be a topological space and $\Lambda \subseteq \mathbb{R}$. A collection of open sets $\{U_\lambda\}_{\lambda\in\Lambda}$ is said to be normally ascending provided that for any $\lambda_1, \lambda_2 \in \Lambda$,

$$\overline{U_{\lambda_1}} \subseteq U_{\lambda_2} \quad \text{if } \lambda_1 < \lambda_2.$$

Example: Let $f: X \to \mathbb{R}$ be continuous and let $\Lambda = \mathbb{R}$.

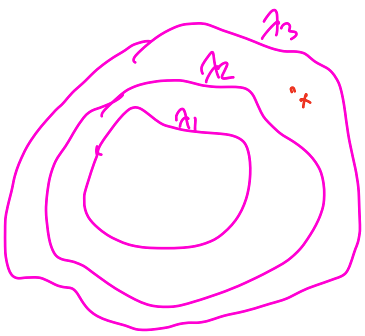
For $\lambda \in \mathbb{R}$, $U_\lambda = f^{-1}(-\infty, \lambda)$.

Then for any $\lambda_1 < \lambda_2$, $\overline{U_{\lambda_1}} \subseteq f^{-1}((-\infty, \lambda_1]) \subseteq f^{-1}(-\infty, \lambda_2) = U_{\lambda_2}$

And so $\{U_\lambda\}_{\lambda\in\Lambda}$ is normally ascending.

Lemma 1: Let X be a topological space. Let $\Lambda \subseteq (0,1)$ be dense in $(0,1]$. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of open sets that are normally ascending. Define the function $f: X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1, & x \notin \bigcup_{\lambda\in\Lambda} U_\lambda \\ \inf\{\lambda\in\Lambda \mid x \in U_\lambda\}, & x \in \bigcup_{\lambda\in\Lambda} U_\lambda \end{cases}$$



Then $f: X \to [0,1]$ is continuous.

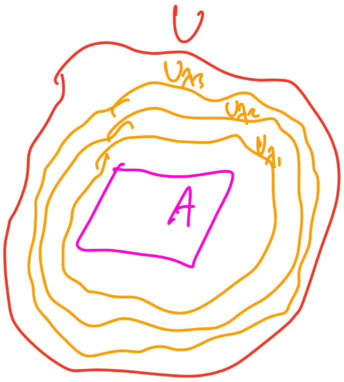
Proof: Let (a,b) be an arbitrary interval.

$$f^{-1}(a,b) = \bigcup_{\substack{\cancel{\lambda\in\Lambda,} \\ \lambda\in\Lambda\cap(a,b)}} U_\lambda \setminus \overline{U_{\max\{a,0\}}}$$ (show this)

(show $U_\lambda = f^{-1}(-\infty,\lambda]$)

Lemma 2: Let X be a normal space. Let A be a closed set and U be a neighbd of A. Then $\exists$ a countable dense set $\Lambda \subseteq (0,1)$ and a normally ascending $\{U_\lambda\}_{\lambda\in\Lambda}$ s.t.

$$A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq U \qquad \forall \lambda \in \Lambda.$$



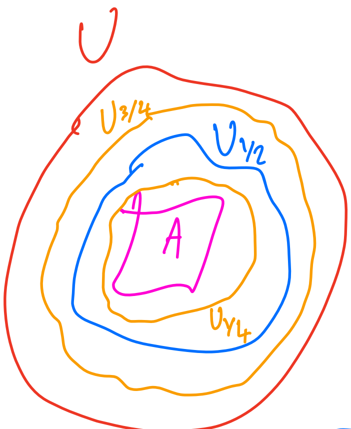
Proof:

Let Λ be the dyadic rationals:

$$\Lambda := \left\{ \frac{m}{2^n} \;\middle|\; m, n \in \mathbb{N} \;\middle|\; m \leq 2^n - 1 \right\}$$

which is dense in $(0,1]$. (show this)

We will inductively define a sequence of collections of normally according open sets $\{U_\lambda\}_{\lambda \in \Lambda_n}$, where $\Lambda_n := \left\{ \frac{m}{2^n} \;\middle|\; \begin{array}{l} m \leq 2^n - 1 \\ m \in \mathbb{N} \end{array} \right\}$



Choose an open set $U_{1/2}$ s.t.

$$A \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U$$

Then we have defined $\{U_\lambda\}_{\lambda \in \Lambda_1}$

Since A is closed and $U_{1/2}$ is a neighbhd of A, we can choose an open set $U_{1/4}$ s.t.

$$A \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2}.$$

Since $\overline{U_{1/2}}$ is a closed set and U is a neighbhd of $\overline{U_{1/2}}$, we can choose an open set $U_{3/4}$ s.t.

$$\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U.$$

So $A \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U$

Then we have extended the normally ascending collection $\{U_\lambda\}_{\lambda\in\Lambda_1}$ to the normally ascending Collection $\{U_\lambda\}_{\lambda\in\Lambda_2}$

We Proceed inductively to define for each $n \in \mathbb{N}$, a normally ascending collection $\{U_\lambda\}_{\lambda\in\Lambda_n}$.

Observe that the union of those collections is a normally ascending collection of open sets Parametrized by Λ.

So $\{U_\lambda\}_{\lambda\in\Lambda}$ is the desired collection. ∎

Now we can Prove Urysohn Lemma!

<u>Proof</u>: Let A and B be disjoint closed sets in a normal top. space X.

Let $U = B^c$, which is a neighbd of A.

Using lemma2, we can find a normally ascending collection of open sets $\{U_\lambda\}_{\lambda \in \Lambda}$

s.t. $\Lambda = \{\text{dyadic rationals}\}$ and $A \subseteq U_\lambda \subseteq \overline{U_\lambda} \subseteq B^c$ $\forall \lambda \in \Lambda$.

Then the continuous function $f: X \to [0,1]$ from lemma1 is the desired function.

$f(B) = \{1\}$ by construction.

If $x \in A$, then $x \in U_\lambda$ $\forall \lambda \in \Lambda$ $\Rightarrow f(x) = \inf\{\lambda \in \Lambda \mid x \in U_\lambda\} = 0$

$\Rightarrow f(A) = \{0\}$

$\blacksquare$

Urysohn lemma is an extension thm:

Let $f: A \cup B \to [0,1]$ defined by $f(A) = \{0\}$ and $f(B) = \{1\}$

which is a continuous function.

Urysohn lemma asserts that $\exists$ a cont extension $\tilde{f} : X \to [0,1]$

<u>Tietze Extension Thm</u>: Let X be a normal space, $A \subseteq X$ be a closed set, and $f : A \to [a,b]$ be a continuous function. Then $\exists$ a cont extension $\tilde{f} : X \to [a,b]$

<u>Proof</u>: Assume wlog that $[a,b] = [-1,1]$

We will construct a sequence of cont functions $g_n : X \to [-1,1]$ satisfying:

for $n \in \mathbb{N}$,
$$|g_n| \leq \left(\frac{2}{3}\right)^n \quad \text{on } X$$
$$\left|f - \sum_{k=1}^{n} g_k\right| \leq \left(\frac{2}{3}\right)^n \quad \text{on } A.$$

<u>Claim</u>: For $a > 0$ and cont function $h : A \to \mathbb{R}$ s.t. $|h| \leq a$ on A, $\exists$ a cont function $g : X \to \mathbb{R}$ satisfying $|g| \leq (1/3)a$ on X and $|h - g| \leq \left(\frac{2}{3}\right)a$ on A.

<u>Proof of Claim</u>: Let $F_1 := h^{-1}[-a, \frac{-1}{3}a]$ and

$$F_2 := h^{-1}\left[\tfrac{1}{3}a, a\right]$$

which are disjoint closed sets.

By Urysohn lemma, there is a continuous function $g: X \to \left[\tfrac{-1}{3}a, \tfrac{1}{3}a\right]$ s.t. $g(F_1) = \left\{\tfrac{-1}{3}a\right\}$ and $g(F_2) = \left\{\tfrac{1}{3}a\right\}$

Then $|g| \leq \left(\tfrac{2}{3}\right)a$ on X and $|h-g| \leq \left(\tfrac{2}{3}a\right)$ on A

show this: take cases $x \in F_1$, $x \in F_2$, $x \notin F_1 \cup F_2$

Apply the claim for $h = f$ and $a = 1$ to find a cont function $g_1 : X \to \mathbb{R}$ satisfying

$$|g_1| \leq \frac{2}{3} \text{ on } X \text{ and } |f - g_1| \leq \frac{2}{3} \text{ on } A.$$

We apply the claim again for $h = f - g_1$ and $a = \frac{2}{3}$ to find a cont function $g_2 : X \to \mathbb{R}$ satisfying

$$|g_2| \leq \left(\frac{2}{3}\right)^2 \text{ on } X \text{ and } |f - g_1 - g_2| \leq \left(\frac{2}{3}\right)^2 \text{ on } A.$$

Proceed inductively to define a sequence $g_n : X \to \mathbb{R}$

of continuous functions satisfying:

* $|g_n| \leq \left(\frac{2}{3}\right)^n$ on X

* $\left|f - \sum_{k=1}^{n} g_k\right| \leq \left(\frac{2}{3}\right)^n$ on A

Then define $\tilde{f} : X \to \mathbb{R}$ by

$\tilde{f}(x) := \sum_{k=1}^{\infty} g_n(x)$ which converges uniformly

and so $\tilde{f}$ is continuous. (There will be a post-lecture practice-question to help you prove this)

and $\tilde{f} = f$ on A since for each $x \in A$, we have that $\left|f(x) - \sum_{k=1}^{n} g_k(x)\right| \leq \left(\frac{2}{3}\right)^n \quad \forall n \in \mathbb{N}$

and hence $|f(x) - \tilde{f}(x)| \leq 0 \implies f(x) = \tilde{f}(x)$

$\therefore$ $\tilde{f}$ is the desired extension. $\blacksquare$

The next Consequence of Urysohn Lemma is a metrization thm

<u>Urysohn Metrization Thm:</u>

Let X be second countable space. Then X is metrizable iff X is normal.

<u>Proof</u>: $(\Rightarrow)$ ✓

$(\Leftarrow)$ Suppose X is normal and second countable.

Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable basis.

Define $A := \{ (n,m) \in \mathbb{N}^2 \mid \overline{U_n} \subseteq U_m \}$

A is nonempty (show this)

For each $(n,m) \in A$, by Urysohn lemma, there is a continuous function $f_{(n,m)} : X \to [0,1]$ s.t.

$f_{(n,m)}(\overline{U_n}) = \{0\}$ and $f_{(n,m)}(U_m^c) = \{1\}$

For $x, y \in X$, define $d(x,y) := \sum_{(n,m)\in A} \frac{1}{2^{n+m}} |f_{(n,m)}(x) - f_{(n,m)}(y)|$

Show that d is indeed a metric.

To show that the topology induced by d is the given topology on X, we need to prove:

① for $x \in U_k$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U_k$

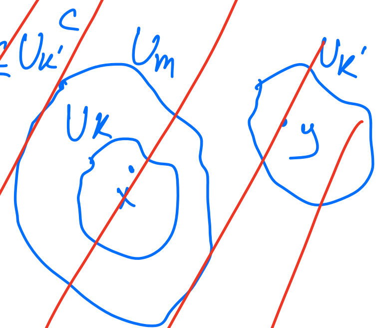
② for $\varepsilon > 0$ and $x \in X$, $U_k \subseteq B_\varepsilon(x)$ for some $k \in \mathbb{N}$.

①: Let $\varepsilon = \frac{1}{2^k}$, then $y \in B_{1/2^k}(x)$

$\Rightarrow \frac{1}{2^{n+m}} |f_{(n,m)}(x) - f_{(n,m)}(y)| < \frac{1}{2^k} \quad \forall n, m \in \mathbb{N}.$

$\Rightarrow f_{(k,m)}(x) = f_{(k,m)}(y)$ and $f_{(n,k)}(x) = f_{(n,k)}(y) \quad \forall n, m \in \mathbb{N}.$

Suppose $y \notin U_k$. Let $m \in \mathbb{N}$ s.t. $\overline{U_k} \subseteq U_m \subseteq U_{k'}^c$



and so $f_{(k,m)}(x) = 0$

and $f_{(k,m)}(y) = 1$ since $y \in U_m^c$

which is a contradiction

$\Rightarrow y \in B_\varepsilon(x) \Rightarrow y \in U_n$ and ① ✓

Check Post-Lecture-Question #4.

② Show This

Check Post-lecture question #5

Another approach is to show that

$f: X \to \mathbb{R}^A$ defined by

$f(x) = \left(f_{(n,m)}\right)_{(n,m)\in A}$ is an embedding

Showing that X is homeomorphic to a subset of $\mathbb{R}^{\mathbb{N}}$ and hence X is metrizable.

Post-lecture Practice Questions

1) Do the exercises above.

2) Let $a, b \in (0,1)$, $a < b$.

a) Show that $\exists n \in \mathbb{N}$ s.t. $2^n(b-a) > 1$

b) Show $\exists m \in \mathbb{N}$ s.t. $m < 2^n(b-a) \leq m+1$

c) Show that $\Lambda = \left\{ \frac{m}{2^n} \,\middle|\, m,n \in \mathbb{N},\ m \leq 2^n - 1 \right\}$ is dense in $(0,1)$.

3) Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions.
We say $f_n \to f$ uniformly if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.
$|f_n(x) - f(x)| < \varepsilon$.

a) Show that if $f_n \to f$ uniformly, then f is continuous.
Hint: Let $(a,b) \subseteq \mathbb{R}$ and let $x \in f^{-1}(a,b)$.
Argue that $\exists N \in \mathbb{N}$ and $\delta > 0$ s.t. $x \in f_N^{-1}\left(f(x)-\delta, f(x)+\delta\right) \subseteq f^{-1}(a,b)$

b) Let $\{a_n\}_{n \in \mathbb{N}}$ be a positive sequence in $\mathbb{R}$ s.t.
$\sum a_n < \infty$.
Let $g_n : X \to \mathbb{R}$ be a sequence of functions s.t.
$|g_n| \leq a_n \quad \forall n \in \mathbb{N}$.
Show that $f_n := \sum_{n=1}^{n} g_n$ converges uniformly to
$f := \sum_{n=1}^{\infty} g_n$. Conclude that f is continuous.

4) We will prove (1) in the proof of Urysohn Metrization thm

a) Let $x \in X$ and let $k \in \mathbb{N}$ s.t. $x \in U_k$.
Suppose $U_k \neq X$ (or else we are done).

Argue that $\exists m \in \mathbb{N}$ s.t. $(m,k) \in A$

b) Let $U = f_{(m,k)}^{-1}\left([0,\frac{1}{2})\right)$. Show that $x \in U \subseteq U_k$

c) For $\varepsilon = \frac{1}{2^{m+k+1}}$. Show that $B_\varepsilon(x) \subseteq U \subseteq U_k$.

5) We will prove ② in the proof of the metrization thm.

Let $\varepsilon > 0$ and $x \in X$. We wish to show that $U_k \subseteq B_\varepsilon(x)$ for some $k \in \mathbb{N}$

a) Let $l \in \mathbb{N}$ s.t. $\frac{1}{2^{2l}} < \frac{\varepsilon}{8}$.

Let $U = \bigcap_{\substack{(m,n) \in A \\ m,n < l}} f_{(m,n)}^{-1}\left([0,\frac{\varepsilon}{2})\right)$ which is open in X.

Show that $U \subseteq B_\varepsilon(x)$

b) Conclude that $U_k \subseteq B_\varepsilon(x)$ for some $k \in \mathbb{N}$.

Let $y \in U$.

6) Write the collection $\{f_{(n,m)}\}_{(n,m)\in A}$ from the Proof of the metrization thm as $\{g_n\}_{n\in\mathbb{N}}$.

Define the map $F: X \longrightarrow \mathbb{R}^{\mathbb{N}}$ by
$$F(x) = (g_n(x))_{n\in\mathbb{N}}.$$

Show that $F(U)$ is open in $F(X)$ whenever U is open. Conclude that F is an embedding and that X is metrizable.

7) Let X be a normal space and let A be a closed set. Let $f: A \to \mathbb{R}$ be a continuous function.

a) Apply the Tietze Extension thm to obtain a continuous extension $\tilde{h}: X \to [0,1]$ of $h = \dfrac{f}{1+|f|}$.

b) Apply the Urysohn lemma to obtain a continuous function $\phi: X \to [0,1]$ s.t. $\phi(A)=\{1\}$ and $\phi(\tilde{h}^{-1}(1))=\{0\}$.

c) Show that $\exists$ continuous extension $\tilde{f}: X \to \mathbb{R}$ of f.
(Consider $\tilde{f} = \dfrac{\phi h}{1-\phi h}$)

8) Solve #3 ~ #5 in Section 34.

9) Solve #3 - #4 in Section 35