* Mistake in last lecture. Stone-Čech compactification of $(0,1)$ is not the topologist's sine curve.

The separation axioms.

$$T_1 \Leftarrow T_2 \Leftarrow T_3 \Leftarrow T_{3.5} \Leftarrow T_4 \Leftarrow T_5$$

T_2: Hausdorff; T_3: regular; $T_{3.5}$: completely regular; T_4: normal; T_5: completely normal

A T_1 space is completely regular if for every $x \in X$ and every closed set A that doesn't contain x, $\exists$ continuous function $f: X \to [0,1]$ s.t. $f(x)=0$ and $f(A)=\{1\}$.

<u>Theorem</u>: Let X be a completely regular space. There exists a compactification Y, called the Stone-Čech compactification of X, with the property that every bounded continuous function $f: X \to \mathbb{R}$ can be uniquely extended to a continuous function on Y into $\mathbb{R}$.

<u>Proof</u>: Let $\{f_\alpha\}_{\alpha \in J}$ be the collection of all bounded cont functions from X to $\mathbb{R}$.

for each $\alpha \in J$, let $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$

We define $h: X \longrightarrow \prod_{\alpha \in J} I_\alpha$ by $h(x) = (f_\alpha(x))_{\alpha \in J}$

We claim that h is an embedding.

⊛ h is continuous since the components f_α are all continuous.

⊛ h is injective: let $x \neq y$. $\exists$ a cont $f: X \to [0,1]$ s.t. $f(x) = 0$ and $f(y) = 1$. Since f is bounded, $f = f_\beta$ for some $\beta \in J$. Since $f_\beta(x) \neq f_\beta(y)$, we have that $(f_\alpha(x))_{\alpha \in J} \neq (f_\alpha(y))_{\alpha \in J}$.

⊛ $h: X \to h(X)$ is open:

Let $U \subseteq X$ be an open set. We want to show $h(U)$ is open in $h(X)$

Let $z_0 \in h(U)$; then $z_0 = (f_\alpha(x_0))_{\alpha \in J}$ for some $x_0 \in U$. Since X is completely regular, $\exists$ a cont function $f: X \to [0,1]$ s.t. $f(x_0) = 0$ and $f(U^c) = \{1\}$

Since f is bounded, $f = f_\beta$ for some $\beta \in J$.

Let $W = \pi_\beta^{-1}([0,1)) \cap h(X)$ which is an open neighbhd of z_0.

Let $z \in W$; then $z = (f_\alpha(x))_{\alpha \in J}$ for some $x \in X$. Since $f_\beta(x) \neq 1$, it follows that $x \in U$ since $f_\beta(U^c) = \{1\}$ and $f_\beta(x) \neq 1$. And so $W \subseteq h(U)$.

$\therefore$ $h: X \to \prod_{\alpha \in J} I_\alpha$ is an embedding. Let Y be the compactification of X induced by h (recall $Y \cong \overline{h(X)}$) and so $\exists$ an embedding $H: Y \to \prod_{\alpha \in J} I_\alpha$ s.t. $H|_X = h$ (due to Thm from last lecture)

We claim that Y is the desired compactification. Let $f_\alpha : X \to \mathbb{R}$ be a bounded continuous function. Then $\tilde{f}_\alpha := \pi_\alpha \circ H$ is the desired extension. $\tilde{f}_\alpha$ is unique due to the following lemma:

<u>Lemma</u>: Let $A \subseteq X$ and let Y be Hausdorff. Let $f : A \to Y$ be a continuous function. Then f has at most one continuous extension $\tilde{f} : \overline{A} \to Y$. (Show this)

$\circledast$

Corollary:

<u>Embedding Thm</u>: Let X be completely regular. Then X can be embedded into $[0,1]^J$ for some J.

<u>Corollary</u>: Let X be completely regular. Let C be any compact Hausdorff space. Let Y be the Stone-Čech Compactificati. Then any cont function $f : X \to C$ can be uniquely extended from Y into C.

<u>Proof</u>: C is compact Hausdorff $\Rightarrow$ C is normal
$\Rightarrow$ C is completely regular by Urysohn's lemma.

So $\exists$ an embedding $h : C \to [0,1]^J$.
Assume wlog that $C \subseteq [0,1]^J$

Let $f : X \to C$ be a continuous function. We can write $f = (f_\alpha)_{\alpha \in J}$ where $f_\alpha : X \to [0,1]$.
Since each f_α is bounded and cont, it can be uniquely extended to a cont function $\tilde{f}_\alpha : Y \to [0,1]$.
Define $\tilde{f} : Y \to C$ by $\tilde{f}(x) := (\tilde{f}_\alpha(x))$ which is the desired extension. (Show that $\tilde{f}$ is cont).

Proposition: The Stone-Čech compactification is unique up to equivalence.

Proof:
Let Y_1 and Y_2 be two compactifications with the property described in the existence thm.

Let $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ be the inclusion maps.
Since Y_1 is a compact Hausdorff space, by corollary $\exists!$ cont extension $f_1 : Y_2 \to Y_1$
Since Y_2 is a compact Hausdorff space, similarly $\exists!$ cont extension $f_2 : Y_1 \to Y_2$.

Then $f_1 \circ f_2 : Y_1 \to Y_1$ satisfies $f_1 \circ f_2|_X = Id|_X$

Since $Id : Y_1 \to Y_1$ is a continuous extension of $f_1 \circ f_2$, then $f_1 \circ f_2 = Id$ by the uniqueness of continuous extensions. (by lemma above). Similarly $f_2 \circ f_1 = Id$

This implies that f_1 and f_2 are homeomorphisms s.t. $f_1|_X = Id$ and $f_2|_X = Id$. We conclude Y_1 and Y_2 are equivalent.

$\blacksquare$

For a completely regular space X, we denote the Stone-Čech compactification by $\beta(X)$.

Proposition: Let X be completely regular.

$\beta(X)$ is the "maximal" compactification of X in the sense that if Y is a compactification of X, then

$\exists$ continuous surjective closed map $g : \beta(X) \to Y$ s.t. $g|_X = Id$.

Prove this

Post-Lecture-Practice-Questions

1) Do the exercises above

2) Let X be completely regular. Show that $\beta(X)$ is connected iff X is connected.

3) Let X be a discrete space.
   a) Show that X is completely regular.
   b) Show that for any $A \subseteq X$, $\overline{A}$ and $\overline{A^c}$ are disjoint, where the closures are taken in $\beta(X)$.
   c) Show that if U is open in $\beta(X)$, then $\overline{U}$ is open in $\beta(X)$.
   d) Show that $\beta(X)$ is totally disconnected

4) Solve #8 and #9 in Munkres section 38.