* Assignment 4 is due today by 5PM.

Separation Axioms

We have studied:

T_1: A space is T_1 if $\forall x, y \in X$ s.t. $x \neq y$, $\exists$ neighbd U of x that doesn't contain y.

T_2: Hausdorff.

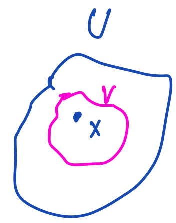
Def: Let X be a T_1 space. X is regular (T_3) if for each pair of disjoint closed sets A and B, where A is a singleton, $\exists$ neighbds of A and B that are disjoint.
Let X be a T_1 space. X is normal (T_4) if for each pair of disjoint closed sets A and B, $\exists$ neighbds of A and B that are disjoint.

Lemma $T_1 \Leftarrow T_2 \Leftarrow T_3 \Leftarrow T_4$

Lemma: Let X be a T_1 space.

a) X is regular iff for any $x \in X$ and neighbd U of x, $\exists$ neighbd V of x s.t. $\overline{V} \subseteq U$.

b) X is normal iff for any closed set A and neighbd U of A, $\exists$ neighbd V of A s.t. $\overline{V} \subseteq U$.



Proof of (a): ($\Longrightarrow$) Suppose X is regular.

Let $x \in X$ and U be a neighbd of x.

Then $\{x\}$ and U^c are disjoint closed sets and so $\exists$ neighbd V of x and V' of U^c that are disjoint.

Since $V \cap V' = \phi$, $V \subseteq V'^c \Rightarrow \overline{V} \subseteq V'^c$

Since $U^c \subseteq V' \Rightarrow V'^c \subseteq U$

($\Longleftarrow$) Let $x \in X$ and A be a closed set that doesn't contain x.

Then A^c is a neighbd of x and so $\exists$ neighbd V of x s.t. $\overline{V} \subseteq A^c$. Then V and $\overline{V}^c$ are disjoint neighbds of $\{x\}$ and A respectively.

Thm: Both Hausdorff and regular are hereditary and productive.

Prove This

<u>Examples:</u> 1) Define $\mathbb{R}_K$ to be $\mathbb{R}$ equipped with the topological space generated by the basis

$$\mathcal{B} = \{(a,b) \mid a<b\} \cup \{(a,b)\setminus K \mid a<b\}$$

where $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

$\mathbb{R}_K$ is finer than $\mathbb{R}$ so it's also Hausdorff.

$\mathbb{R}_K$ is not regular. The set K is closed in $\mathbb{R}_K$ and $0 \notin K$ but $\{0\}$ and K cannot be separated by disjoint by open sets. (fill in the details)

2) $\mathbb{R}_\ell$ is normal. Singletons are closed in $\mathbb{R}_\ell$ since $\mathbb{R}_\ell$ is finer than $\mathbb{R}$. Suppose A and B are disjoint closed sets.

For $a \in A$, let $[a, x_a)$ be disjoint from B.

for $b \in A$, let $[b, x_b)$ be disjoint from A.

Then $U = \bigcup_{a\in A} [a, x_a)$ and $V = \bigcup_{b \in B} [b, x_b)$ are the desired open sets. (Show that U and V are disjoint)

However $\mathbb{R}_\ell^2$ is NOT normal.

That shows a) regular $\not\Rightarrow$ normal.

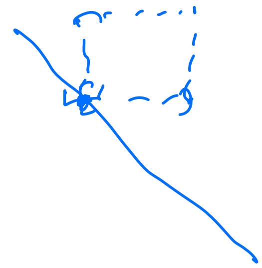
b) normal is not finitely productive.

Prove:

Let $L = \{(x,-x) \in \mathbb{R}_\ell^2 \mid x \in \mathbb{R}\} \subseteq \mathbb{R}_\ell^2$

L is closed in $\mathbb{R}^2_\ell$ and has the discrete topology.

Then for any $A \subseteq L$, A and $L \setminus A$ are closed in L and so in $\mathbb{R}^2_\ell$.



Suppose $\mathbb{R}^2_\ell$ is normal.
Then for every nonempty proper subset A of L, $\exists$ disjoint neighbds U_A and V_A of A and $L \setminus A$.

Let $D = \mathbb{R}^2_\ell \cap \mathbb{Q}$ which is dense in $\mathbb{R}^2_\ell$.

We define a map $F : P(L) \to P(D)$ by

$$F(A) = \begin{cases} U_A \cap D & , A \neq \phi, L \\ \phi & , A = \phi \\ D & , A = L \end{cases}$$

Then F is injective. Show this.

$\Rightarrow |P(\mathbb{R})| = |P(L)| \leq |P(D)| = |\mathbb{R}|$

contradiction.

☒

What guarantees Normality.

Theorem 1: Every Second countable regular space is normal.

Proof: Let $\mathcal{B}$ be a countable basis.
Let A and B be disjoint closed sets.

For each $a \in A$, $\exists$ neighbd U_a of a s.t. $\overline{U_a} \subseteq B^c$.
For each $b \in B$, $\exists$ neighbd V_b of b s.t. $\overline{V_b} \subseteq A^c$

For each a and b, pick a basis neighbd of a and b that is contained in U_a and V_b respectively.

Let $\{U_n \mid n \in \mathbb{N}\}$ and $\{V_n \mid n \in \mathbb{N}\}$ be the collection of neighbds that we defined.

$U := \bigcup_{n \in \mathbb{N}} U_n$ and $V := \bigcup_{n \in \mathbb{N}} V_n$ are neighbds of

A and B but are not necessarily disjoint.

Define $U_n' = U_n \setminus \left(\bigcup_{k=1}^{n} \overline{V_n} \right)$
$V_n' = V_n \setminus \left(\bigcup_{k=1}^{n} \overline{U_n} \right)$ } both open

Then $U' := \bigcup_{n\in\mathbb{N}} U_n'$ and $V' := \bigcup_{n\in\mathbb{N}} V_n'$ are the desired neighbds. Show a) U' and V' are neighbds of A and B respectively.

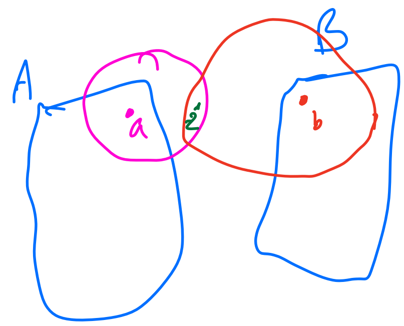
b) $U' \cap V' = \phi$

$\blacksquare$

Theorem 2: Every metrizable space is normal.

Proof: Let (X,d) be a metric space. Let A and B be disjoint closed sets.

For each $a\in A$, $\exists\, \varepsilon_a > 0$ s.t. $B_{\varepsilon_a}(a) \subseteq B^c$
For each $b \in B$, $\exists\, \varepsilon_b > 0$ s.t. $B_{\varepsilon_b}(b) \subseteq A^c$



Let $U := \bigcup_{a\in A} B_{\frac{\varepsilon_a}{2}}(a)$ and $V := \bigcup_{b\in B} B_{\frac{\varepsilon_b}{2}}(b)$

Then U and V are neighbds of A and B respectively.

We show that U and V are disjoint.

Suppose $\exists z \in B_{\varepsilon_a/2}(a) \cap B_{\varepsilon_b/2}(b)$.

Then $d(a,b) < \frac{\varepsilon_a + \varepsilon_b}{2}$ by the triangle inequality

$\leq \varepsilon_b$ assuming wlog $\varepsilon_b \geq \varepsilon_a$

$d(a,b) < \varepsilon_b \Rightarrow a \in B_{\varepsilon_b}(b)$ which is a contradiction.

$\blacksquare$

Theorem: Every compact Hausdorff space is normal.

Proven in the past.

Theorem (Urysohn lemma):

Let X be a normal space. Let A and B be disjoint closed subsets of X.

$\exists$ a continuous function $f: X \to [0,1]$

s.t. $f(x) = 0 \ \forall x \in A$ and $f(x) = 1 \ \forall x \in B$

($A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$)

Def: If A and B are subsets of a topological space X and if $\exists$ a continuous $f: X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$, we say that A and B can be separated by a continuous function.

Def: A T_1 space X is completely regular ($T_{3.5}$) if for each $x \in X$ and each closed set A not containing x, $\exists$ continuous function $f: X \to [0,1]$ s.t. $f(x) = 1$ and $f(A) = \{0\}$.

Lemma: $T_{3.5} \Rightarrow T_3$. $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ are the desired neighbds.

Stone - Čech Compactification

* A compactification of a topological space is a compact Hausdorff space Y s.t. X is a subspace and $\overline{X} = Y$.

* If X is noncompact locally compact Hausdorff,

Then it admits a unique one point compactification, which is a compactification Y of X s.t. $Y \setminus X$ is a singleton.

* If X has a compactification, then X is regular.

* $f: X \to Y$ is an embedding if f is homeomorphic onto its image.

Lemma: Suppose $f: X \to Z$ is an embedding of X into a compact Hausdorff space Z. Then $\exists$ a compactification Y of X; it has the property that there is an embedding $F: Y \to Z$ s.t. $F|_X = f$. The compactification Y is uniquely determined up to equivalence.

(Recall: Y and Y' are equivalent compactifications of X if $\exists$ homeomorphism $g: Y \to Y'$ s.t. $g|_X = Id$)

We call Y the compactification induced by the embedding f.

Prove this

Ex: (1) Let $f:(0,1)\to S^1$ defined by $f(t)=(\cos 2\pi t, \sin 2\pi t)$.
Then f is an embedding and S^1 is the compactification induced by f, which is equivalent to the one-point compactification of $(0,1)$.

(2) $X=(0,1)$. Then $Y=[0,1]$ is another compactification of X that distinct from the one above.

(3) Let $h:(0,1)\to\mathbb{R}^2$ defined by
$h(x)=(x, \sin\frac{1}{x})$ which is an embedding.
Then $\overline{h(0,1)}$ = Topologist's sine curve is the compactification of $(0,1)$ induced by h.

Question: If Y is a compactification of X, under what conditions can a continuous real-valued function on X be extended continuously to Y?

A necessary condition on f is that it has to be bounded.

For ex (1): Sufficient Conditions: $*$ f is bounded
and necessary
$*$ $\lim_{x\to 0^+} f = \lim_{x\to 1^-} f$ (2)

→ ② Sufficient and necessary condition : * f is bounded

* $\lim_{x\to 0^+} f$ and ①
$\lim_{x\to 1^-} f$ need to exist.

③ Sufficient and necessary condition : * f is bounded

* more...

But collection of functions that can be extended contains ① and ②

Remark:

($f(x) = \sin\frac{1}{x}$ on $(0,1)$ can be extended continuously to its 3rd compactification despite the fact that $\lim_{x\to 0^+} f$ DNE).

Theorem: let X be a completely regular space. $\exists$ a compactification of X having the property that every bounded continuous map $f : X \to \mathbb{R}$ can be extended uniquely to a continuous function $\tilde{f} : Y \to \mathbb{R}$.

This compactification is Stone - Čech compactification

~~(The Stone - Čech compactification for $(0,1)$ is the topologist's sine curve).~~

Post-lecture-Practice-Questions.

1) Solve the exercises above
2) Define an equivalence relation on $[0,1]$ as follows: $x \sim y$ if $x-y \in \mathbb{Q}$.
   Show that $[0,1]/\sim$ is not Hausdorff.

3) Show every metric space is completely regular. (Don't use Urysohn Lemma)

4) Let $X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. Define a basis $\mathcal{B}$ for a topology on X as follows:
   * for $(a,b) \in X$ with $b > 0$, $B_\varepsilon(a,b) \in \mathcal{B}$ for $0 < \varepsilon < b$
   * for $(a,0) \in X$, $\{(a,0)\} \cup B_\varepsilon(a,\varepsilon) \in \mathcal{B}$.

   X is called the Moore plane

   a) Show X is completely regular.

   b) Show X is not normal and hence not metrizable.
   Hint: let $A = \{(a,0) \mid a \in \mathbb{Q}\}$ and $B = \{(a,0) \mid a \notin \mathbb{Q}\}$

5) Let $X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(0,-1)\}$. Define a basis $\mathcal{B}$ for a topology on X as follows.
   * If $(x,y) \in X$ and $y > 0$, then $\{(x,y)\} \in \mathcal{B}$
   * for each $x \in \mathbb{R}$, let $M_x = \{(x,y) \mid 0 \leq y < 2\}$, $N_x = \{(x+y,y) \mid 0 \leq y < 2\}$

and P_x be a finite subset of $(M_x \cup N_x) \setminus \{(x,0)\}$.
Then $(M_x \cup N_x) \setminus P_x \in \mathcal{B}$
Show that X is regular but not completely regular.

6) Show that Completely regular is abitrarily Productive and hereditary.

7) Let A be a compact subset of a regular space and let U be a neighbd of A. Show $\exists$ a neighbd V of A s.t. $\overline{V} \subseteq U$.

8) A T_1 space is normal iff for each Pair of disjoint closed sets A and B, $\exists$ neighbds U and V of A and B respectively s.t. $\overline{U} \cap \overline{V} = \phi$.

9) Every Closed Subset of a normal space is normal.

10) Let $f: (0,1) \to \mathbb{R}$ be a bounded continuous function.
Let $Y_1 = S^1$, $Y_2 = [0,2]$, $Y_3 =$ topologist's Sine curve be three compactifications of $(0,1)$.
a) Show Y_1, Y_2, Y_3 are not equivalent

b) Show f can be extended to Y_1 iff $\lim_{x\to 0^+} f = \lim_{x\to 1^-} f$

c) Show f can be extended to Y_2 iff $\lim_{x\to 0^+} f$ and $\lim_{x\to 1^-} f$ exist.

d) Show f can be extended to Y_3 if $\lim_{x\to 0^+} f$ and $\lim_{x\to 1^-} f$ exist.

e) Show $f(x) = \sin\frac{1}{x}$ can be extended to Y_3 but $f(x) = \cos\frac{1}{x}$ cannot be extended to Y_3. Conclude that Y_3 is not the Stone-Čech compactification of X.

f) Find another compactification Y_4 of $(0,1)$ s.t. all bounded continuous function s.t. $\lim_{x\to 0^+} f$ & $\lim_{x\to 1^-} f$ exist can be extended as well as $f(x) = \sin\frac{1}{x}$ and $f(x) = \cos\frac{1}{x}$.