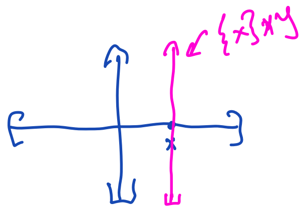
* Deadline for Ass 4 is pushed to Wednesday 5PM and Problem 4 is removed.
* Reading from last week (skip parts we didn't cover)

Tychonoff Thm

How did we prove $X_1 \times X_2$ is compact if X_1, X_2 are compact?



* Begin with an open cover for $X_1 \times X_2$
* find a finite subcover for $\{x\} \times Y$ for each $x \in X$ as $\{x\} \times Y \cong Y$ is compact.
* Construct a finite subcover for $X \times Y$.

Alternate approach: We will use

X is compact $\Longleftrightarrow$ Every collection of closed sets with the finite intersection property has a nontrivial intersection.

Let's first use this proof to prove that $X_1 \times X_2$ is compact whenever X_1 and X_2 are compact.

Let $\mathcal{A}$ be a collection of closed sets in $X_1 \times X_2$ with the finite intersection property.

Then $\mathcal{A}_1 = \{ \overline{\pi_1(A)} \mid A \in \mathcal{A} \}$ and $\mathcal{A}_2 = \{ \overline{\pi_2(A)} \mid A \in \mathcal{A} \}$ are collections of closed sets in X_1 and X_2 respectively with the finite intersection property.

By compactness of X_1 and X_2, $\exists\, x_1 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_1(A)}$ and $x_2 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_2(A)}$.

Will $(x_1, x_2) \in \bigcap_{A \in \mathcal{A}} A$? Not necessarily.

One way to think about it $\{ \overline{\pi_1(A)} \times \overline{\pi_2(A)} \mid A \in \mathcal{A} \} \nsubseteq \mathcal{A}$.

Also, elements in A are not of the form $A_1 \times A_2$ where A_1 and A_2 are closed. (Exc: if all the elements in $\mathcal{A}$ are indeed of this form, then $(x_1, x_2) \in \bigcap_{A \in \mathcal{A}} A$).

Think of an example in $\mathbb{R}^2$

Heuristic Argument:

If $\mathcal{A}$ is a collection of sets with the finite intersection property s.t. it's "as big as possible".

Let $x_1 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_1(A)}$ $x_2 \in \bigcap_{A \in \mathcal{A}} \overline{\pi_2(A)}$.

Let U_1 be any neighbd of x_1, then $U_1 \times X_2$ intersects every set in $\mathcal{A}$. Since $\mathcal{A}$ is as big as possible,

$U_1 \times X_2 \in \mathcal{A}$.

Similarly $X_1 \times U_2 \in \mathcal{A}$ for every neighbd U_2 of x_2 in X_2.

$U_1 \times U_2 = \left(X_1 \times U_2\right) \cap \left(U_1 \times X_2\right)$ that also must be an element in $\mathcal{A}$.

In Particular $U_1 \times U_2$ intersects A for every $A \in \mathcal{A}$.

$\Rightarrow (x_1, x_2) \in \overline{A} \quad \forall A \in \mathcal{A}$.

<u>Def</u>: Let X be a set. let $\mathcal{D}$ be a collection of subsets of X. We say $\mathcal{D}$ is maximal wrt the finite intersection property if

1) $\mathcal{D}$ has the finite intersection property
2) No collection of subsets of X that property contains $\mathcal{D}$ has the finite intersection property.

Lemma 1: Let X be a set. Let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property. There is a collection $\mathcal{D}$ that contains $\mathcal{A}$ and is maximal wrt finite intersection property.

Lemma 2: Let $\mathcal{D}$ be a collection of subsets of X that is maximal wrt finite intersection property.

1) If $A \subseteq X$ that intersects every element in $\mathcal{D}$, then $A \in \mathcal{D}$.
2) Any finite intersection of elements in $\mathcal{D}$ is in $\mathcal{D}$.

Proof

Lemma 3: Let $X = \prod_{\alpha \in J} X_\alpha$.

Let $\mathcal{D}$ be a collection of subsets of X that is maximal wrt finite intersection property. Suppose $\bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$ is nonempty $\forall \alpha \in J$.

Let $(x_\alpha)_{\alpha\in J} \in X$ s.t. $x_\alpha \in \bigcap_{D\in\mathcal{D}} \overline{\pi_\alpha(D)}$ for $\alpha \in J$.

Then $(x_\alpha)_{\alpha\in J} \in \bigcap_{D\in\mathcal{D}} \overline{D}$.

$U_\alpha \cap \pi_\alpha(D) \neq \emptyset \quad \forall D \in \mathcal{D}$

Proof:

Let $\alpha \in J$. Let U_α be an arbitrary neighbd of x_α in X_α. Since ~~$x_\alpha \in \overline{\pi_\alpha(D)}\ \forall D \in \mathcal{D}$~~, we have that $\pi_\alpha^{-1}(U_\alpha) \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

And so by lemma 2, $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{D}$.

Since every basis neighborhood of $(x_\alpha)_{\alpha\in J}$ is an intersection of finitely many sets of the form $\pi_\alpha^{-1}(U_\alpha)$ where U_α is a neighbd of x_α, it follows by lemma 2 that every basis neighbd is in $\mathcal{D}$.

$\Rightarrow$ Every basis neighbd of $(x_\alpha)_{\alpha\in J}$ intersects every element in $\mathcal{D}$

$\Rightarrow \quad (x_\alpha)_{\alpha\in J} \in \overline{D} \quad \forall D \in \mathcal{D}$.

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Tychonoff Theorem:

Compactness is arbitrarily productive.

Proof: Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of compact spaces. Let $X = \prod_{\alpha \in J} X_\alpha$.

Let $\mathcal{A}$ be a collection of closed sets with the finite intersection property.

By lemma 1, $\exists$ a collection $\mathcal{D}$ of subsets of X containing $\mathcal{A}$ and is maximal wrt the finite intersection property.

By compactness of X_α, the collection $\{\overline{\pi_\alpha(D)} \mid D \in \mathcal{D}\}$

have a nontrivial intersection for each $\alpha \in J$.

Let $(x_\alpha)_{\alpha \in J} \in X$ s.t. $x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$

Then by lemma 3, $(x_\alpha)_{\alpha \in J} \in \overline{D}$ $\forall D \in \mathcal{D}$

$$\Rightarrow (x_\alpha)_{\alpha\in J} \in \overline{A} = A \quad \forall A \in \mathcal{A}$$

And so $\mathcal{A}$ has a nontrivial intersection.
We conclude X is compact

Post-Lecture-Practice-Questions

1) Do the above exercises.

2) Let $\mathcal{A}$ be the collection of all closed elliptical regions bounded by ellipses in $[0,1]^2$ that have the point $p=(\frac{1}{3},\frac{1}{3})$ and $q=(\frac{1}{2},\frac{2}{3})$ as their foci.

 a) Verify that $\mathcal{A}$ has the finite intersection property. Find the intersection of all the sets in $\mathcal{A}$.

 b) Find a point $(x,y)\in[0,1]^2$ s.t. $x\in\bigcap_{A\in\mathcal{A}}\overline{\pi_1(A)}$ and $y\in\bigcap_{A\in\mathcal{A}}\overline{\pi_2(A)}$ but $(x,y)\notin\bigcap_{A\in\mathcal{A}} A$.

 c) Show that $\mathcal{A}$ is not maximal wrt the finite intersection property by finding a set $B\subseteq[0,1]^2$ s.t. $\mathcal{A}\cup\{B\}$ has the finite intersection property.

3) Solve #7 in Munkres Ch 37.

4) We will prove Lemma 1. Let X be a set and let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property.

Define $\mathcal{E} := \{ \mathcal{B} \supseteq \mathcal{A} \mid \mathcal{B}$ has the finite intersection property $\}$

a) Convince yourself that $\subsetneq$ defines a strict partial order on $\mathcal{E}$ meaning:

1) $\mathcal{B} \subsetneq \mathcal{B}$ never holds.

2) $\mathcal{B}_1 \subsetneq \mathcal{B}_2$ and $\mathcal{B}_2 \subsetneq \mathcal{B}_3 \Rightarrow \mathcal{B}_1 \subsetneq \mathcal{B}_3$

b) Let $\mathcal{E}' \subseteq \mathcal{E}$ be a simply ordered subset of $\mathcal{E}$, meaning $\forall \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{E}'$ either $\mathcal{B}_1 = \mathcal{B}_2$, $\mathcal{B}_1 \subsetneq \mathcal{B}_2$ or $\mathcal{B}_2 \subsetneq \mathcal{B}_1$.

Show that $\mathcal{E}'$ has an upperbound in $\mathcal{E}$, meaning $\exists \mathcal{B} \in \mathcal{E}$ s.t. $\forall \mathcal{B}' \in \mathcal{E}'$, either $\mathcal{B}' = \mathcal{B}$ or $\mathcal{B}' \subsetneq \mathcal{B}$.

c) Recall Zorn's lemma:

Let $\mathcal{E}$ be a strictly partially ordered set. If every simply ordered set of $\mathcal{E}$ has an upperbound in $\mathcal{E}$, then $\mathcal{E}$ has a maximal element, which is an element $\mathcal{B} \in \mathcal{E}$ s.t. there doesn't exist $\mathcal{B}' \in \mathcal{E}$ satisfying $\mathcal{B} < \mathcal{B}'$.

Conclude the proof of Lemma 1 by invoking Zorn's lemma on $(\mathcal{E}, \subsetneq)$.