* Required readings from last week

* Mistake in proof of
"countably compact ⊕ first countable $\Rightarrow$ sequentially compact"

Thm: Let X be a metric space. Then X is compact iff X is complete and totally bounded.

Proof: ($\Longrightarrow$) X is compact $\Rightarrow$ X is sequentially compact (since X is first countable)

$\Rightarrow$ X is complete.

↳ let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. Then it has a subsequence that converges which implies that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Let $\varepsilon>0$ Then $\{B_\varepsilon(x)\}_{x\in X}$ is an open cover has a finite subcover and so X is totally bounded.

($\Longleftarrow$) Suppose X is complete and totally bounded. We will prove X is sequentially compact. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X.

We will construct a converging sequence. (Assume wlog that $x_n \neq x_m$ for $n\neq m$)

Cover X with finitely many balls with radius 1, one of which must contain infinitely many elements of the sequence. Call it B_1

Let $J_1 := \{n\in\mathbb{N} \mid x_n \in B_1\}$ which is infinite.

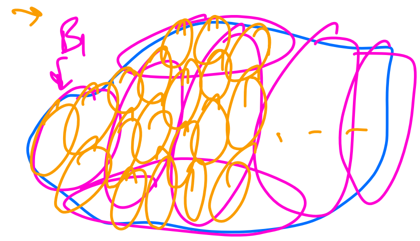
Cover X with finitely many balls with radius $1/2$, one of which must contain infinitely many x_n's where $n\in J_1$.

Call it B_2. Let $J_2 := \{n \in \mathbb{N} \mid x_n \in B_2\} \cap J_1$

Continue this procedure recursively.

Then $\{J_n\}_{n\in\mathbb{N}}$ are all infinite and nested in the sense that $J_n \supseteq J_{n+1}$ $\forall n \in \mathbb{N}$.

Construct a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ s.t. $n_k \in J_k$.



Then for any $m, k \in \mathbb{N}$ $d(x_{n_k}, x_{n_{k+m}}) < \frac{1}{k}$ since $n_k, n_{k+m} \in J_k$ implying $x_{n_k}, x_{n_{k+m}} \in B_k$, which is a ball of radius $\frac{2}{k}$. This implies $\{x_{n_k}\}_{k\in k}$ is a Cauchy sequence in X. By completeness, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges.

$\boxtimes$

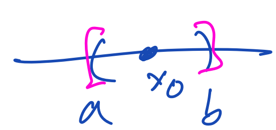
This is a generalization of Heine borell.

local compactness & Compactification

<u>Def</u>: X is locally compact at $x \in X$ if there is some compact set C that contains a neighborhood of x.

X is locally compact if it's locally compact at every point.

<u>Ex</u>:
- Any compact set is locally compact.
- $\mathbb{R}$ is locally compact.
- Similarly, $\mathbb{R}^n$ is locally compact



(*) $\mathbb{R}^{\mathbb{N}}$ is not locally compact. Take any basis set $U := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_k, b_k) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$

Notice $\overline{U} = [a_1, b_1] \times \cdots \times [a_k, b_k] \times \mathbb{R} \times \mathbb{R} \times \cdots$ is not compact and so U cannot be contained in a compact set.

Theorem: Let X be a Hausdorff space. Then X is locally compact iff $\forall x \in X$, $\exists$ neighbd U of x s.t. $\overline{U}$ is compact.

Proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ Let $x \in X$.

Since X is locally compact, $\exists$ neigbd U of x and a compact set C s.t. $U \subseteq C$.

$$\Rightarrow \overline{U} \subseteq \overline{C} = C$$

since X is Hausdorff

Since $\overline{U}$ is a closed subset of a compact set C, $\overline{U}$ is compact.

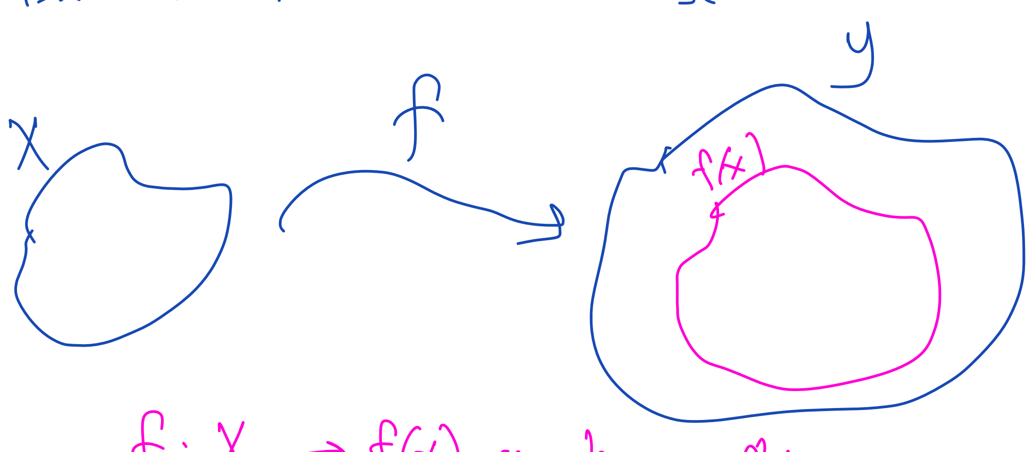
∎

Two important types of topological spaces: ① Metric spaces.
② Compact Hausdorff spaces.

The next best thing is if X is a subset of one of these two topological spaces.

Def: Let X be a Hausdorff space. We say Y is a compactification of X if Y is a compact Hausdorff space that contain X as a subspace and $\overline{X} = Y$.
A one-point compactification of X is a compactification Y s.t. $Y \setminus X$ is a singleton.

Def: An embedding $f: X \to Y$ is a function that is a homeomorphism onto its image.



$f: X \to f(X)$ is a homeomorphism.

Def: We say that the compactifications Y and Y' of X are equivalent if $\exists$ a homeomorphism $f: Y \to Y'$ s.t. $f|_X = Id$.

Observation: Let Y be a compact Hausdorff space. Let $p \in Y$ s.t. $X := Y \setminus \{p\}$ is noncompact. Then X is a locally compact Hausdorff space with Y as a one-point compactification.

(show X is locally compact)

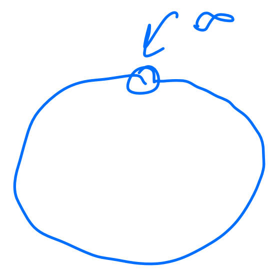
In fact.

Theorem: X is a noncompact locally compact Hausdorff space iff it admits a one-point compactification. Furthermore, the one-point compactification is unique up to equivalence.

Proof: ($\Leftarrow$) ✓

($\Rightarrow$) Let X be a non compact locally compact

Hausdorff space.



Let ∞ be a point not in X and let $Y = X \cup \{\infty\}$.

$$\text{Define } \mathcal{T} = \{U \subseteq X \mid U \text{ is open in } X\} \cup \{U \subseteq Y \mid \infty \in U \text{ and } U^c \text{ is compact in } X\}$$

We want to show $\circledast$ $\mathcal{T}$ is indeed a topology on Y

on Assignment 4. $\to$ $\circledast$ $(Y, \mathcal{T})$ is compact and Hausdorff.

$\circledast$ $\overline{X} = Y$

$\Longrightarrow$ Y is a one-point compactification of X.

Show Y is unique up to equivalence.

$$\left(\text{Let } f : \overset{Y}{X \cup \{\infty\}} \longrightarrow \overset{Y'}{X \cup \{\infty'\}} \quad f|_X = \mathrm{Id} \text{ and } f(\infty) = \infty' \right)$$

Post lecture Practice Questions.

1) Do the exercises above

2) Use one-point compactification Theorem to prove the following:

   a) A Hausdorff space is locally compact iff $\forall x \in X$ and for every neighbd U of x, $\exists$ neighbd V of x s.t. $V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact

   b) Let X be locally compact and Hausdorff. If $A \subseteq X$ is open in X, then A is locally compact. (use (a)).

3) Show that every closed subspace of a locally compact Hausdorff space is locally compact.

4) $\prod_{\alpha \in J} X_\alpha$ is locally compact iff X_α is locally compact $\forall \alpha \in J$ and X_α is compact for all but finitely many $\alpha \in J$.

5) Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and let $\mathcal{B} = \{U \subseteq \mathbb{R} \mid U$ is an open interval not containing $0\}$

$\cup \{(-x,x) \setminus A \mid x > 0\}$

Show that $\mathcal{B}$ is a basis for a topology τ on $\mathbb{R}$ s.t. $(\mathbb{R}, \tau)$ is not locally compact and A is closed.

6) Let X be connected, locally connected, locally compact Hausdorff space. Let $x, y \in X$. Show that $\exists$ compact connected subset of X containing both x and y.

7) Solve 1, 4, 5, 6, 8 in Ch 29 Munkres.