* Tutorials & OH for Jiawei
* Mistake in last lecture regarding Lebesgue #.

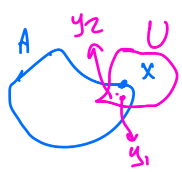
You can attempt to prove that limit point compact $\Rightarrow$ countably compact, but you will get stuck. (You will need a separation axiom).

<u>Def</u>: A topological space X is T_1 if $\forall x, y \in X$, $x \neq y$, $\exists$ neighbd U of x s.t. $y \notin U$.

$\Longleftrightarrow$ Singletons are closed. Show this.

<u>Lemma</u>: Let X be a T_1 topological space. Let x be a limit point of a set $A \subseteq X$. Then every neighbd of x intersects A at infinitely many points.

Proof



$\rightarrow X \setminus \{y_1, y_2\} \cap U$ is another neighbd of x . . .

$X \setminus \{y_1\} \cap U$ is another neighbd of x and so intersects A at a point y_1

<u>Proposition 4</u>: Let X be a T_1 topological space. Then X is limit point compact iff it's countably compact.

Proof: $\Rightarrow$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a nested sequence of nonempty closed sets.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence s.t. $x_n \in A_n$ for $n\in\mathbb{N}$. Define $A := \{x_n | n\in\mathbb{N}\}$

Since X is limit point compact, there exists a limit point y of A.

Then every neighbd of y intersects A at infinitely many points, and hence intersects A_n for every $n\in\mathbb{N}$ $\Rightarrow y \in \overline{A_n} = A_n \ \forall n\in\mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} A_n$

Prop 5: Sequentially compact $\Rightarrow$ Countably Compact.

Proof: Let $\{A_n\}_{n\in\mathbb{N}}$ be a nested sequence of nonempty closed sets.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence s.t. $x_n \in A_n$ for $n\in\mathbb{N}$. Define $A := \{x_n | n\in\mathbb{N}\}$

Since X is sequentially compact, $\exists$ subsequence x_{n_k} that converges to $y \in X$.

Then every neighbd of y intersects A at infinitely many points and $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ from the previous proof.

$\boxtimes$

Proposition 6: Let X be a first countable space.

X is countably compact iff it's sequentially compact.

1st countable + limit point compact

*) sequentially

Proof: ($\Leftarrow$) by Prop 5

($\Rightarrow$): Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X.

Assume wlog that $x_n \neq x_m$ for $n \neq m$.

~~Since X is countably compact, it's also limit point compact. Since $A := \{x_n \mid n\in\mathbb{N}\}$ is infinite, $\exists$ limit point x of A. Since X is first countable, $\exists$ a sequence $\{y_k\}_{k\in\mathbb{N}}$ in A s.t. $y_k \to x$. Since $y_k \in A$, $y_k = x_{n_k}$ for some n_k and so $\{y_k\}_{k\in\mathbb{N}} = \{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges.~~

Define $A_k := \{x_n \mid n > k\}$.

Suppose $\{x_n\}_{n\in\mathbb{N}}$ has no converging subsequence. Then $\{\overline{A_k}\}_{k\in\mathbb{N}}$ is a nested collection of nonempty closed sets. Since X is countably compact, $\bigcap_{k=1}^{\infty} \overline{A_k} \neq \emptyset$. Let $x \in \bigcap_{k\in\mathbb{N}} \overline{A_k}$. Every neighbd of x intersects A_k $\forall k\in\mathbb{N}$. Contradiction.

Proposition 7: Let X be second countable space.

Then X is sequentially compact $\Rightarrow$ X is compact.

Proof: Let $\mathcal{U}$ be an open cover. Since X is second countable, $\mathcal{U}$ admits a countable subcover $\{U_n\}_{n\in\mathbb{N}}$.

(show this)

Suppose $\nexists$ finite subcover. Construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ s.t. $x_1 \in U_1$ and $x_{n+1} \notin \bigcup_{i=1}^{n} U_i$

Since X is sequentially compact, $\exists x_{n_k}$ that converges to x.
Suppose $x \in U_m$ for $m \in \mathbb{N}$. Since U_m is a neighbd of x,
$\exists N \in \mathbb{N}$ s.t. $x_{n_k} \in U_m$ $\forall k > N$, which is a contradiction

$\boxtimes$

<u>lemma</u>: A sequentially compact metric space is second countable.

<u>Proof</u>:

<u>Theorem</u>: Let X be a metric space. Then the following are equivalent.

1) X is compact
2) X is sequentially compact
3) X is countably compact
4) X is limit point compact.

second countable & metric space

first count.

T_1

follows from lemma + Prop - 7

<u>Theorem</u>: Let X be a metric space.
Then X is compact iff it's complete & totally bounded.

Summary

$$\begin{tikzcd}
\text{Compact} \arrow[dr] & & \text{Sequentially Compact} \arrow[ll, swap, "X \text{ is second countable}"] \arrow[dl, shift left] \\
& \text{Countably compact} \arrow[ur, shift left, swap, "X \text{ is first countable}"] \arrow[d, shift right] & \\
& \text{Limit point Compact} \arrow[u, shift right, swap, "X \text{ is } T_1"] &
\end{tikzcd}$$

Post-Lecture-Practice-Questions

1) Do the exercises above.

2) The following are the steps to prove the above lemma. Suppose X is a sequentially compact metric space. We want to show that X is second countable.

   a) Apply some of the propositions we proved earlier to argue that X is totally bounded.

b) In light of (a), for every $n \in \mathbb{N}$ there exists a finite set $F_n = \{x_1, \dots, x_k\}$ s.t. the ϵ-balls centered at $x_i \in F_n$ form a cover for X. Show that $D = \bigcup_{n=1}^{\infty} F_n$ is a countable dense set. Conclude that X is separable and hence second countable.

3) Let X be a metric space. X is compact iff it's complete and totally bounded.

Use a theorem proven in today's lecture as well as Propositions 1-7 to prove the forward direction.

4) Prove that every compact subset of a metric space is closed and bounded.

5) Give an example of a compact metric space X and an non-Hausdorff space Y and a continuous surjective function $f: X \to Y$.

6) Prove that continuous functions send countably compact sets to countably compact sets

Is the same true for sequentially compact?

7) Prove that sequentially compact is countably productive.

8) This is generalization of the closed map lemma:

Let X be countably compact, Y be first countable.

Let $f: X \to Y$ be a continuous bijection. Show f is a homeomorphism