* Assignment 4 will be posted soon.

Different definitions of compactness

Def: X is **limit point compact** if every infinite set has a limit point.

Def: X is **countably compact** if every countable open cover admits a finite subcover.

Def: X is **sequentially compact** if every sequence has a converging subsequence.

Def: A collection of closed sets $\{A_\alpha\}_{\alpha\in J}$ has the finite intersection property if every finite subcollection has a non-trivial intersection.

Proposition: X is compact iff every collection of closed sets with the finite intersection property has a non-trivial intersection.

Proof: Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of closed sets.

Let $U_\alpha = A_\alpha^c$. Then $\bigcap_{\alpha \in J} A_\alpha = \phi \iff \bigcup_{\alpha \in J} U_\alpha = X$.

Continue from there. ∎

Proposition: X is countably compact iff every countable collection of closed sets with the finite intersection property has a non-trivial intersection iff every nested sequence $\{A_n\}_{n \in \mathbb{N}}$ of nonempty closed sets satisfies $\bigcap_{n=1}^{\infty} A_n \neq \phi$.

$(A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots)$

Proof

Recall by Heine Borel: $A \subseteq \mathbb{R}^n$ is compact iff it's closed and bounded. (wrt d_{euc}) We would like to generalize this thm to arbitrary metric spaces.

$\bar{d}(x,y) = \min\{1, d_{euc}(x,y)\}$

Observation 1: $(\mathbb{R}^n, \bar{d})$ is topologically equivalent to $(\mathbb{R}^n, d_{euc})$, but $\mathbb{R}^n$ is closed and bounded wrt $\bar{d}$ but not compact.

Def: Let (X, d) be a metric space. X is totally bounded if $\forall \varepsilon > 0$ $\exists$ finite covering of X by ε-balls.

Exc: Let d_1, d_2 be equivalent metrics on X. Then X is totally bounded wrt d_1 iff the same holds wrt d_2.

Exc: X is totally bounded $\Rightarrow$ X is bounded.

on $\mathbb{R}^n$, $A \subseteq \mathbb{R}^n$ is totally bounded $\Leftrightarrow$ A is bounded wrt to d_{euc}.

Exc: Show $(\mathbb{R}^n, \bar{d})$ is bounded but not totally bounded.

Observation 2: Consider $X = \mathbb{R}^n \setminus \{0\}$. Let B be the closed ball of radius 1 in X. B is closed & totally bounded but not compact.

Def: Let (X, d) be a metric space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon$ whenever $n, m > N$.

We say (X,d) is complete if every Cauchy sequence converges.

Exc: Any closed subset of a complete metric space is complete. Also a complete subset of a metric space is closed.

Proposition 1: Compact $\Rightarrow$ limit point compact.

Proof: Let A be an infinite set. Suppose A has no limit points. For every $x \in X$, let U_x be a neighborhood s.t. $U_x \cap A = \{x\}$ or ϕ. Since $\{U_x\}_{x\in X}$ is an open cover, $\exists$ a finite subcover $U_{x_1}, \cdots, U_{x_n}$. Since $U_{x_i} \cap A = \{x_i\}$ or ϕ and $\bigcup_{i=1}^{n} U_{x_i} = X$, it follows that $A \subseteq \{x_1, \cdots, x_n\}$ contradiction. $\square$

Proposition 2: Let (X,d) be a limit point compact metric space. Then X is totally bounded.

Proof: Suppose X is not totally bounded.

Then $\exists\, \varepsilon > 0$ s.t. X cannot be covered by finitely many ε-balls. We will construct sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying $\{x_n | n\in\mathbb{N}\}$ has no limit point.

Let $x_1 \in X$. Then $B_\varepsilon(x_1) \subsetneq X$. Let $x_2 \in X \setminus B_\varepsilon(x_1)$.
Then $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \subsetneq X$. Let $x_3 \in X \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$
and so on. This defines a sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying
$d(x_n, x_m) \geq \varepsilon$ for $n \neq m$ $\Rightarrow$ $\{x_n \mid n \in \mathbb{N}\}$ is an infinite set without a limit point, which is a contradiction. ☒

called Lebesgue # for $\mathcal{A}$

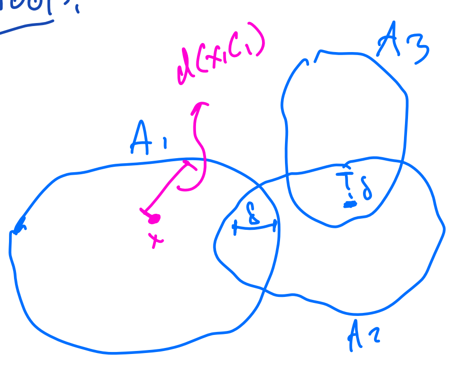
<u>Lebesgue # lemma</u>: Let X be a compact metric space and let $\mathcal{A}$ be an open cover. $\exists\, \delta > 0$ s.t. each subset with diameter $< \delta$ is contained in an element in $\mathcal{A}$.

$\operatorname{diam}(B) = \sup_{x,y \in B} d(x,y)$

no Lebesgue # for $\mathcal{A} = \{(-1,1)\}$

~~$(-1,1) \subseteq \mathbb{R}$~~

<u>Proof</u>:



Since X is compact, $\exists$ finite subcover $A_1, \ldots, A_n$.

Let $C_i = A_i^c$.

(If $X = A_i$ for some $1 \leq i \leq n$, then any positive # is a Lebesgue #)
Assume that $A_i \neq X$ $\forall\, 1 \leq i \leq n$

Note that the function $x \longmapsto d(x, C_i)$ is cont.
(exc)

Define $f: X \to (0, \infty)$ by
$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$
We first show $f > 0$. Let $x \in X$, then $x \in A_i$ for some $1 \le i \le n$.
So $\exists \varepsilon > 0$ $B_\varepsilon(x) \subseteq A_i$ $\Rightarrow d(x,y) \ge \varepsilon$ $\forall y \in C_i$

$\Rightarrow d(x, C_i) \ge \varepsilon$

$\Rightarrow f(x) > 0$ since $d(x, C_j) \ge 0$ for all $1 \le j \le n$ & $d(x, C_i) > 0$.

Since $f > 0$ and cont on a compact set, by EVT it follows that $f(x) \ge \inf_{x \in X} f = f(x_{min}) =: \delta > 0$

$\Rightarrow f(x) \ge \delta \quad \forall \, x \in X$

If B is a set with diameter $< \delta$, then B is contained in the ball $B_\delta(x)$ for some $x \in B$. Let $d(x, C_k) = \max_{1 \le i \le n} \{ d(x, C_i) \}$
> 0

Since $d(x, C_k) \ge f(x) > \delta \Rightarrow B \subseteq B_\delta(x) \subseteq A_k$

$\boxtimes$

Ex: Lebesgue # lemma holds also for arbitrary limit point compact metric spaces.

Def: Let $f: (X, d_X) \to (Y, d_Y)$. We say f is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x,y) < \delta$.

Ex: Uniformly cont $\Rightarrow$ cont.

Uniform Continuity Thm: Let $f: (X, d_X) \to (Y, d_Y)$ be a continuous function on a compact metric space X. Then f is uniformly cont.

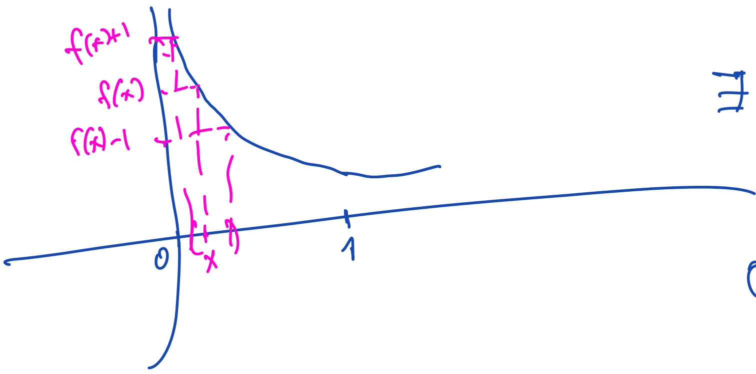
Proof: Let $\varepsilon > 0$. For $x \in X$, let $U_x := f^{-1}(B_{\varepsilon/2}(f(x)))$ which is open since f is cont.

Then $\{U_x\}_{x \in X}$ is an open cover for the compact set X and so admits a Lebesgue # δ by the Lebesgue # lemma. Let $x, y \in X$ satisfying $d_X(x,y) < \delta$. Then $\{x,y\}$ has diameter $< \delta$ and so $\{x,y\} \subseteq U_{x_0}$ for some $x_0 \in X$. And so

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_0)) + d_Y(f(x_0), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, f is uniformly cont. $\blacksquare$

Ex: $f(x) = \frac{1}{x}$ on $(0,1)$

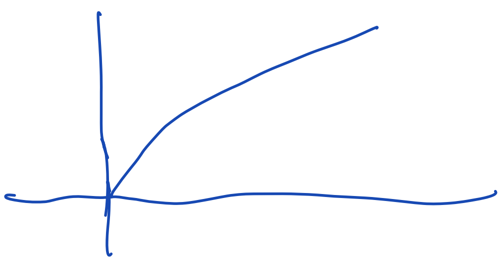


Choose $\varepsilon = 1$. Let $x \in (0,1)$

$\exists\, \delta_x$ s.t. $|x-y| < \delta \Rightarrow$
$\left|\frac{1}{x} - \frac{1}{y}\right| < 1$

Observe $\inf_{x \in (0,1)} \delta_x = 0$

Ex: $f(x) = \sqrt{x}$ on $(0,1)$



f is uniformly cont since f can be extended to a continuous function on $[0,1]$ & so by the uniform continuity thm, f is uniformly cont on $[0,1]$

$\Rightarrow$ f is uniformly cont on $(0,1)$

But f is not Lipschitz.

Ex: $f(x) = x \sin\frac{1}{x}$ on $(0,1]$

f is uniformly cont since $\tilde{f}(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a continuous extension of f on

The compact set $[0,1]$

On the other hand, $f(x) = \sin\frac{1}{x}$ is cont on $(0,1]$ but not uniformly cont! (But it's uniformly cont on $(\varepsilon, 1]$ for any $\varepsilon > 0$)

<u>Proposition 3</u>: Compact $\overset{①}{\Rightarrow}$ countably compact $\overset{②}{\Rightarrow}$ limit point compact.

<u>Proof</u>: ① is trivial.

②: Let A be an infinite set. Assume wlog that A is countable, so $A = \{x_n \mid n \in \mathbb{N}\}$.

Then define the sets $C_k := \{x_n \mid n > k\}$.

Suppose A has no limit point, so $\{C_k\}_{k=1}^{\infty}$ is a nested collection of nonempty closed sets. ($C_k \supseteq C_{k+1}$ $\forall k \in \mathbb{N}$)

Since X is countably compact, $x_m \in \bigcap_{k=1}^{\infty} C_k$ for some $m \in \mathbb{N}$ which is a contradiction since $x_m = x_n$ for infinitely many $n \in \mathbb{N}$.

$\blacksquare$

Post-Lecture-Practice-Questions.

#1) Do the exercises above

#2) Let X be a totally bounded metric space. Show that X is separable. Conclude that any compact metric space is second countable.

#3) Let $\mathcal{B}$ be a basis for X. Show that X is compact iff every open cover by members in $\mathcal{B}$ has a finite subcover.

#4) Solve #2 on pg 177.

#5) Let $X = \{p, q\}$ be equipped with the indiscrete topology.
Show $\mathbb{N} \times X$ is limit point compact but not compact.

#6) Solve #1, 2, 3 on pg 181

#7) $(\mathbb{R}^{\mathbb{N}}, D)$ is complete where $D(x, y) = \sup_{i \in \mathbb{N}} \left(\frac{\bar{d}(x_i, y_i)}{i} \right)$

#8) $(\mathbb{Q}, d_{euc})$ is not complete