Let's discover more properties of compact spaces:

<u>Theorem</u>: Let $f: X \to Y$ be a continuous map from a compact space to a Hausdorff space. If f is bijective, Then it's a homeomorphism.

<u>Proof</u>: We claim that f is a closed map. Let $A \subseteq X$ be closed and so it's compact. So $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed. & So f is a closed map and hence a homeomorphism.

(f is closed $\Leftrightarrow$ f is open $\Leftrightarrow$ f^{-1} is cont).

$\square$

<u>Example</u>: You defined in 4C an equivalence relation on $[0,1]$ making $[0,1]/\sim$ a quotient space.

You need to show $[0,1]/\sim$ is homeomorphic to S^1.

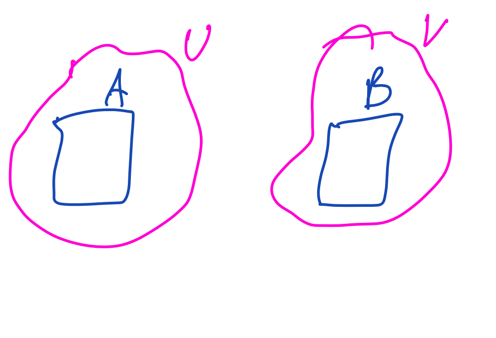
Define a map $f: [0,1] \to S^1$ that is continuous surjective.

$\hookrightarrow$ $\tilde{f}: [0,1]/\sim \to S^1$ that is cont & bij.

Then use the above Thm.

<u>Theorem</u>: Let X be a Hausdorff, Then for any disjoint compact subsets A and B, $\exists$ neighbd U of A and a neighbd V of B s.t. $U \cap V = \phi$.

(disjoint compact subsets can be separated by open subsets)



Proof: Suppose first $B = \{p\}$.

For every $q \in A$, $\exists$ neighbd U_q of q and a neighbd V_q of p s.t. $U_q \cap V_q = \phi$ (using that X is Hausdorff).

Then $\{U_q\}_{q \in A}$ is an open cover for A and so A can be covered by finitely many $U_{q_1}, \dots, U_{q_n}$. Then $V = \bigcap_{i=1}^{n} V_{q_i}$ is a neighbd of p. So $U = \bigcup_{i=1}^{n} U_{q_i}$ and V are the desired neighbds.

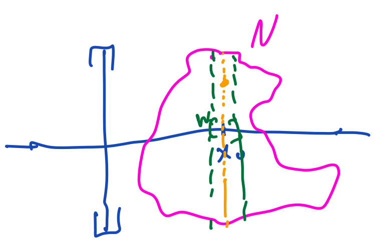
Prove this for the general case

$\blacksquare$

Theorem: Finite product of compact sets is compact.

Tube Lemma: Let X, Y be topological spaces with Y compact. Suppose $x_0 \in X$ and N is an open set containing $\{x_0\} \times Y$. Then $\exists$ neighbd W of x_0 in X s.t.

$$W \times Y \subseteq N.$$



Proof: For $y \in Y$, let $V_y \times U_y$ be a neighbd of (x_0, y) s.t. $V_y \times U_y \subseteq N$.

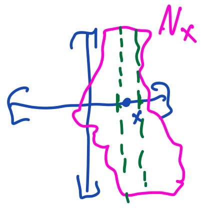
Then $\{U_y\}_{y \in Y}$ is an open cover for Y and hence Y can be covered by finitely many $U_{y_1}, \dots, U_{y_n}$.

Let $W = \bigcap_{i=1}^{n} V_{y_i}$ which is a neighbd of x_0. So

$$W\times Y \subseteq \bigcup_{i=1}^{n}(W\times U_{a_i}) \subseteq N .$$

∎

Proof of Thm: Let X and Y be compact sets. Let $\mathcal{A}$ be an open cover for $X\times Y$. Let $x\in X$.



Then $\{x\}\times Y$ is compact and so it can be covered by finitely many elements in $\mathcal{A}$, say $A_1,\dots,A_m$. Let $N_x := \bigcup_{i=1}^{m} A_i$. Then by the tube lemma, $\exists$ neighbd W_x of x in X s.t. $W_x\times Y \subseteq N_x$.

Since $\{W_x\}_{x\in X}$ is an open cover for X, then X can be covered by finitely many $W_{x_1},\dots,W_{x_n}$.

Since each $W_{x_i}\times Y$ can be covered finitely many elements in $\mathcal{A}$, $X\times Y = \bigcup_{i=1}^{n} W_{x_i}\times Y$ can be covered by finitely many elements in $\mathcal{A}$.

∎

Heine-Borel Thm: A subset of $\mathbb{R}^n$ (w.r.t euclidean metric.) is compact iff it's closed and bounded.

Proof: ($\Longrightarrow$) Let $A\subseteq \mathbb{R}^n$ be compact. Then it's closed because it's a compact subset of a Hausdorff space. A is also bounded because it can be covered by finitely many balls (why?)

($\Leftarrow$) Let A be closed & bounded. Since it's bounded, it's contained in $[-N,N]^n$ for some large $N \in \mathbb{N}$. Since $[-N,N]^n$ is compact by the above thm and A is a closed subset of a compact set, A is compact.

$\boxtimes$

Extreme Value Thm: If X is compact and $f: X \to \mathbb{R}$ is continuous, then f is bounded & attains its maximum and minimum values in X.

Proof: $f(X)$ is compact subset of $\mathbb{R}$ since f is cont. & X is compact. By the above, $f(X)$ is closed and bounded. In particular, $\sup_{x \in X} f(x)$ and $\inf_{x \in X} f(x)$ exist, and since they are both limit points of the closed $f(X)$, they are contained in $f(X)$.

$\boxtimes$

Not on the test.

Different definitions of compactness

Def: X is said to be limit point compact if every infinite subset of X has a limit point in X.

Def: X is said to be sequentially compact if every sequence of points in X has a subsequence that converges in X.

~~Def: X is said to be countably compact if every open cover admits a countable subcover.~~

wrong Def. See next lecture.

We will prove later on that in a metric space X, the following are equivalent:

i) X is compact

ii) X is countably compact

iii) X is limit point compact

iv) X is sequentially compact

v) X is complete and $\underbrace{\text{totally bounded}}_{\text{defined later.}}$

Def: A subset A of a metric space X is totally bounded if $\forall \varepsilon > 0$, $\exists$ finitely many balls with radius ε with center in A covering A.

Exercise: Consider $(\mathbb{R}, \bar{d})$ where $\bar{d}(x,y) := \min\{1, |x-y|\}$.
Show that $\mathbb{R}$ is bounded wrt $\bar{d}$ but it's not totally bounded.

Post-lecture Practice Questions

1) Do the exercises above.
2) Give an example of a bounded metric space that is not compact.
3) Show every compact metric space is second countable.
4) Suppose X is compact with open connected components. Show that there are finitely many components.
5) Is the finite/arbitrary union of compact sets compact?
   Is the finite/arbitrary intersection of compact sets compact?
6) Let $X \times Y$ be a topol. space with Y compact. Show that the projection $\pi : X \times Y \longrightarrow X$ is closed.

7) Let X, Y be topological spaces.

For $f: X \to Y$, we say f is perfect if it's closed continuous surjection with the property that $f^{-1}(y)$ is compact.

Let $f: X \to Y$ be a perfect map.

a) Show that if X is Hausdorff, then so is Y

b) Show that if X is second countable, then so is Y

c) Show that if Y is compact, then so is X.