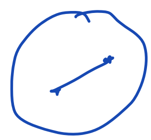
* Assignment 3 is posted & due Sunday 11:59 PM.
* Email me if you cannot make the midterm.

We defined Path-connected and have shown that path-connected $\Rightarrow$ connected.

Examples: ⊛ Open balls in $\mathbb{R}^n$ are path-connected.

Show this.



⊛ $\mathbb{R}^n \setminus \{0\}$ for $n > 1$ is path-connected.

⊛ $S^{n-1}(r) = \{x \in \mathbb{R}^n \mid \|x\| = r\}$ is path-connected where $r > 0$.

Lemma: Continuous functions map path-connected sets to path-connected sets. Prove.

Define $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}(r)$ by $g(x) = \frac{rx}{\|x\|}$

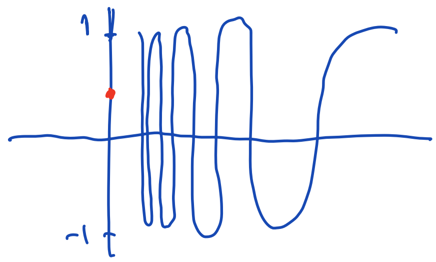
which is a surjective continuous function $\Rightarrow S^{n-1}(r)$ is path-connected

⊛ Define $S = \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\}$

which is connected and path-connected as it's the image of $f : (0,1] \to \mathbb{R}^2$
$: x \longmapsto (x, \sin\frac{1}{x})$

Then $\bar{S}$ is connected where $\bar{S} = S \cup (\{0\} \times [-1,1])$

↳ called the topologist's sine curve.



Claim: S is not Path connected.

Suppose $\exists$ Path $f : [0,1] \to S$ from a point in $\{0\} \times [-1,1]$ to S.

Assume wlog $f(t) \in S \;\; \forall t > 0$. (Argue we can assume this)

Argue by IVT that you can find a sequence $t_n \to 0$ s.t. the y-component of $f(t_n)$ is $(-1)^n$.

Check Post-Lecture-practice-questions.

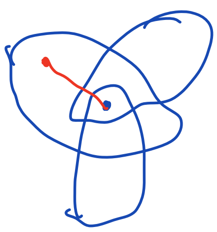
~~Define the sequence~~ $\cancel{t_n = \frac{2}{(2n+1)\pi}}$ ~~which converges to 0.~~

~~Then~~ $\cancel{\sin\frac{1}{t_n} = (-1)^n}$ ~~and so~~ $\cancel{f(t_n)}$ ~~doesn't converge~~

Contradicting the continuity of f.

So Path-connected is strictly stronger than connected.

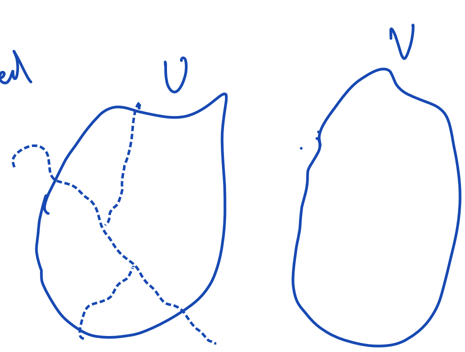
Thm: Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of path-connected sets s.t. $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in J} A_\alpha$ is path-connected.



Proof

Connected & Path-Connected Components

X is not connected



Def: We define an equivalence relation on X as follows: $x \sim y$ if $\exists$ connected (path-connected) subspace containing x and y. The equivalence classes form a partition of X and are called the connected (path-connected) components.

Show that the relations above are indeed equivalence relations.

The following follow directly:

- Ⓧ X is connected (path-connected) iff X has only one connected (path-connected) component.
- Ⓧ Let $x \in X$. Let C_x be the union of all connected (path-connected) sets containing x. Then C_x is the connected (path-connected) component containing x.
- Ⓧ If C is a connected (path-connected) component, then $C = C_x$ $\forall x \in C$.
- Ⓧ Connected (path-connected) components are connected (path-connected)
- Ⓧ Since $\overline{C_x}$ is a connected set containing x, then $\overline{C_x} \subseteq C_x$
  $\Longrightarrow$ Connected components are closed.
- Ⓧ If X has finitely many connected components, then the connected components are also open and hence clopen. (C^c is a finite union of closed sets & C is open)
  (Take the $\mathbb{Q}$ as an example of a space with connected components that are not open)
- Ⓧ Path connected components are not necessarily open nor closed.
  (Exercise: What are the path-connected components of the topologist's sine curve?)

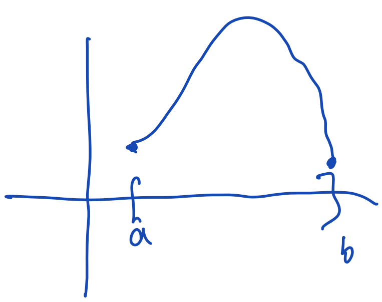
S and $\{0\}\times[-1,1]$

* Every path-connected component is a subset of a connected component.
* If X is locally connected (defined in Assign. 3), then the connected components are open.
  (show C_x is the union of all open connected sets containing x)

* If X is locally path-connected, then the path-connected components are the same as the connected components.

Compactness

EVT:



Doesn't only depend on the continuity of f but also on a certain property of [a,b]

It took a long time to discover that the relevant property of [a,b] to EVT is topological & it took longer to formulate it simply using only open sets.

↳ That Property is very Similar to finiteness & it's Characterized by it's ability to allow us to turn an infinite collection of open sets into a finite subcollection that basically does the same thing. Compact sets can be very large but in a strong sense they behave like a finite set.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function.

Any continuous function is locally bounded. Let $x \in [a,b]$. By choosing $\varepsilon = 1$, we know $\exists \delta > 0$ s.t. $\forall y \in (x-\delta, x+\delta) \cap [a,b]$, $f(y) \in (f(x)-1, f(x)+1)$. And so f is bounded on $(x-\delta, x+\delta) \cap [a,b]$ by $M_x := |f(x)|+1$.

For $x \in [a,b]$, let U_x be a neighborhood of x in $[a,b]$ s.t. f is bounded on U_x by M_x.

Then $\{U_x\}_{x\in[a,b]}$ is a collection of open sets covering $[a,b]$.

If we can find a finite subcollection of $\{U_x\}_{x\in[a,b]}$ that also covers $[a,b]$, then we have shown that f is bounded.

<u>Def</u>: * Let X be a topological space. A collection $\{U_\alpha\}_{\alpha\in J}$ of open sets that cover X (i.e. $\bigcup_{\alpha\in J} A_\alpha = X$) is called an open cover of X.

* X is said to be compact if every open cover of X has a finite subcover (i.e. $\exists$ a finite subcollection of the open cover that covers X).

Example: * $\mathbb{R}$ is not compact. Since $\{(n,n+2)\}_{n\in\mathbb{Z}}$ is a open cover that has no finite subcover.

* Finite sets are always compact.

* $X=\{0\}\cup\{\frac{1}{n} \mid n\in\mathbb{N}\}$ is compact.

Let $\{A_\alpha\}_{\alpha\in J}$ be an open cover for X.

$\exists\, \alpha_0\in J$ s.t. $0\in A_{\alpha_0}$. Since A_{α_0} is open, it contains $\frac{1}{n}$ $\forall\, n>N$ for some $N\in\mathbb{N}$. Find $\alpha_1,\dots,\alpha_N$ s.t. $\frac{1}{m}\in A_{\alpha_m}$ for $1\le m\le N$ and so $A_{\alpha_0}, A_{\alpha_1},\dots, A_{\alpha_N}$ is a finite subcover.

* $(0,1)$ is not compact.

Proof: Consider $\{(\frac{1}{n},1) \mid n\in\mathbb{N}\}$ which is an open cover of $(0,1)$ that has no finite subcover.

Theorem: $[0,1]$ is compact.

Proof: Let $\mathcal{U}=\{U_\alpha \mid \alpha\in J\}$ be an open cover for $[0,1]$

We define $B:=\{s\in[0,1] \mid [0,s] \text{ can be covered by finitely many elements in } \mathcal{U}\}$

Clearly $0 \in B$ since $0 \in U_{\alpha_0}$ for some $\alpha_0 \in J$ and
$[0, \varepsilon) \subseteq U_{\alpha_0}$ for some $\varepsilon > 0$ $\Rightarrow$ $[0, \frac{\varepsilon}{2}] \subseteq B$.
Since B is bounded, it has a supremum denoted by b.

<u>Claim 1</u>: $b \in B$.

Let $\alpha_b \in J$ s.t. $b \in U_{\alpha_b}$. Then $(b-\delta, b+\delta) \cap [0,1] \subseteq U_{\alpha_b}$ for some $\delta > 0$.
By definition of supremum, $\exists s \in (b-\delta, b]$ s.t. $s \in B$ and
so $[0, s]$ can be covered by finitely many elements in $\mathcal{U}$, say
$U_{\alpha_1}, \dots, U_{\alpha_n}$. Then $[0, b]$ can be covered by $U_{\alpha_1}, \dots, U_{\alpha_n}, U_{\alpha_b}$
& so $b \in B$.

<u>Claim 2</u>: If $t_1 \in B$, $t_1 < 1$, then $\exists t_2 \in (t_1, 1]$ s.t. $t_2 \in B$.

Suppose $t_1 \in B$, then $[0, t_1]$ can be covered by $U_{\alpha_1}, \dots, U_{\alpha_n}$
for $\alpha_1, \dots, \alpha_n \in J$. Suppose $t_1 \in U_{\alpha_1}$, then $\exists \varepsilon > 0$ s.t.
$(t_1 - \varepsilon, t_1 + \varepsilon) \cap [0, 1] \subseteq U_{\alpha_1}$ and so pick $t_2 = t_1 + \frac{\varepsilon}{2}$.

<u>Claim 3</u> $b = 1$

Suppose $b < 1$. By Claim 1, $b \in B$. By claim 2,
$\exists t \in (b, 1]$ s.t. $t \in B$ contradicting that $b = \sup B$.

$\blacksquare$

Lemma: Let $Y \subseteq X$ be a subspace. Then Y is compact iff every open cover of Y by sets open in X contain a finite subcollection covering Y.

Proof

Proposition: Let X be a compact set and let $f : X \to Y$ be a continuous function. Then $f(X)$ is compact.

Proof: Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of $f(X)$ where U_α is open in Y.

Then $\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ is an open cover of X

& so admits a finite subcover since X is compact. This implies that $\exists$ finite subcover of $\{U_\alpha\}_{\alpha \in J}$ that covers $f(X)$ and so $f(X)$ is compact.

Theorem: Every closed subspace of a compact is compact.

Proof: Let $Y \subseteq X$ be a closed subspace and X be compact. Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover for Y where U_α is open in X.

Then $\{U_\alpha \mid \alpha \in J\} \cup \{Y^c\}$ is an open cover for X that admits a finite subcover. $\Rightarrow \{U_\alpha \mid \alpha \in J\}$ has a finite subcover for Y.

Theorem: Every compact subspace of a Hausdorff space is closed.

Proof: Let $Y \subseteq X$ be a compact subspace of a Hausdorff space X.

For every $x \in X \setminus Y$. For every $y \in Y$, $\exists$ neighborhoods U_y and V_y of y and x respectively s.t. $U_y \cap V_y = \emptyset$. Then $\{U_y\}_{y \in Y}$ is an open cover for Y that admits a finite subcover $U_{y_1}, \dots, U_{y_n}$.

Then $\bigcap_{i=1}^{n} V_{y_i}$ is an open set containing x that doesn't intersect Y & so $X \setminus Y$ is open $\Rightarrow Y$ is closed

$\boxtimes$

Post - lecture - Practice - Questions

1) Do the exercises above
2) We will fill the gaps in the proof that the topologist's sine curve is not path connected.

Let $f : [0,1] \to \bar{S}$ be a path s.t. $f(0) = (0,0)$ and $f(1) = (1, \sin 1)$

a) Show that $\exists\, s_1, s_2 \in (0,1]$ s.t. $f(s_1) \in \{0\} \times [-1,1]$ and $f(t) \in S$ for $t \in (s_1, s_2]$. Hence, we can assume wlog that $f(0) \in \{0\} \times [-1,1]$ and $f(t) \in S$ $\forall t > 0$

b) Write $f(t) = (x(t), y(t))$ where $x, y : [0,1] \to \mathbb{R}$ are continuous functions satisfying
* $x(0) = 0$, $x(1) = 1$
* $x(t) > 0$ for $t > 0$
* $y(t) = \sin \frac{1}{x(t)}$ for $t > 0$.

Use IVT to argue that $\exists$ sequence $t_n \to 0$ s.t. $x(t_n) = \frac{2}{(2n+1)\pi}$.

c) Conclude that $\bar{S}$ is not path-connected.

2) What are the connected & path connected components of $\mathbb{Q}$? $\mathbb{R}_\ell$? $\mathbb{R}^{\mathbb{N}}$?

3) Consider $(\mathbb{R}^{\mathbb{N}}, \tau_{unif})$. Show that $y \in C_x$ (the connected component containing x) iff the sequence $x-y$ is bounded.

How many components does $(\mathbb{R}^{\mathbb{N}}, \tau_{unif})$ have?

4) Consider $(\mathbb{R}^{\mathbb{N}}, \tau_{box})$ Show that $y \in C_x$ iff $y-x$ is eventually zero.

5) Is path-connected (finitely, countably, arbitrarily) productive?

6) show $\mathbb{R}^2 \setminus$ a countable set is path connected.

7) Consider $\mathbb{Q} \subseteq (\mathbb{R}, \tau_{co\text{-}finite})$. Show $\mathbb{Q}$ is compact but not closed.

8) Show that X is compact iff for every collection $\{A_\alpha\}_{\alpha \in J}$ of closed sets with the property that $\bigcap_{\alpha \in I} A_\alpha \neq \phi$ for every finite $I \subseteq J$, we have that $\bigcap_{\alpha \in J} A_\alpha \neq \phi$.

9) Show that if $f: X \to Y$ is cont, X is compact and Y is Hausdorff, Then f is a closed map (maps closed sets to closed sets)

10) Let $Y_1, Y_2 \subseteq X$ be disjoint compact sets of the Hausdorff space X. Then $\exists$ disjoint sets U, V s.t. $Y_1 \subseteq U$ and $Y_2 \subseteq V$.

11) Let $f: X \to Y$ be a map and Y be compact and Hausdorff. Then f is continuous iff the graph of f $\Gamma_f := \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.