* Mistakes last lecture corrected on Piazza.
* Email me if you cannot make the midterm.

Theorem: $\mathbb{R}$ and intervals are connected. (on Piazza)

Lemma: If $Y \subset \mathbb{R}$ is connected and $x, y \in Y$ s.t. $x < y$, then $[x,y] \subseteq Y$.

Proof: Suppose $\exists\, c \in [x,y] \setminus Y$. Then $(-\infty, c) \cap Y$ and $(c, \infty) \cap Y$ form a separation of Y contradicting the connectedness of Y.

Theorem: The only nonempty connected subsets of $\mathbb{R}$ are intervals. (Prove this)

We proved last lecture:

Theorem: Let X be connected. Let $f: X \to Y$ be a continuous function. Then $f(X)$ is connected.

IVT: Let X be connected and let $f: X \to \mathbb{R}$ be continuous. Let $a, b \in X$ and $r \in \mathbb{R}$ s.t. $f(a) < r < f(b)$. Then $\exists\, c \in X$ s.t. $f(c) = r$.

Proof: Since X is connected, $f(X)$ is a connected subspace of $\mathbb{R}$. By lemma, since $f(a) < f(b)$, $[f(a), f(b)] \subseteq f(X)$. In particular, $\exists\, c \in X$ s.t. $f(c) = r$. ∎

<u>Theorem for subspaces:</u> Let $Y \subseteq X$. The following are equivalent.

a) Y is not connected.

b) $\exists$ disjoint sets A and B whose union is Y, neither of which contains a limit point of the other
(i.e. $\bar{A} \cap B = \phi = A \cap \bar{B}$)

c) $\exists$ open sets U and V in X s.t. $U \cap Y \neq \phi$, $V \cap Y \neq \phi$, $U \cap V \cap Y = \phi$ and $Y \subseteq U \cup V$

<u>Proof:</u> a $\Leftrightarrow$ c

a $\Rightarrow$ b Let A and B form a separation of Y.

$$A = \text{closure of } A \text{ in } Y = \bar{A} \cap Y$$

since A is closed in Y

closure in X

and so $\phi = A \cap B = (\bar{A} \cap Y) \cap B = \bar{A} \cap B$

Similarly, $A \cap \bar{B} = \phi$.

(b $\Rightarrow$ a)

Prove this

Hint: let $U = \bar{A} \cap Y$, $V = \bar{B} \cap Y$ $\Big\}$ form a separation of Y

$\blacksquare$

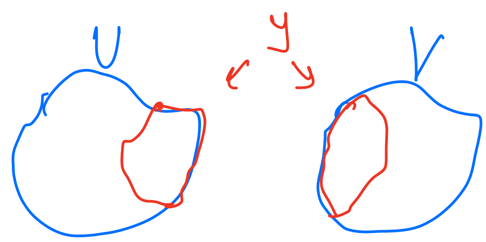
↳ Corollary: Let $Y \subseteq X$ be a connected subspace of X.

Then $\overline{Y}$ is connected. Furthermore, any $Y' \subseteq X$ satisfying $Y \subseteq Y' \subseteq \overline{Y}$ is connected.

Proof (use $a \Leftrightarrow b$ in the above Thm)

Is the product of connected sets connected?

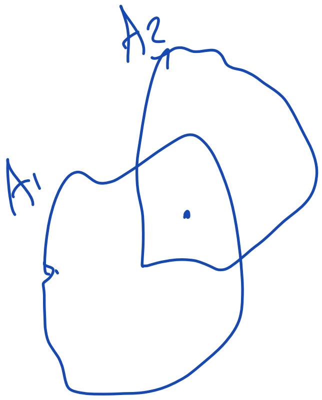
Lemma 1: If sets U and V form a separation of X and Y is a connected subspace of X, then $Y \subseteq U$ or $Y \subseteq V$.



Proof: If not, then $Y \cap U$ and $Y \cap V$ will form a separation of Y contradicting that Y is connected.

Lemma 2: Let $\{A_\alpha\}_{\alpha \in J}$ be connected subsets of X.

If $\bigcap_{\alpha\in J} A_\alpha \neq \phi$, then $\bigcup_{\alpha\in J} A_\alpha$ is connected.



Proof: Let U and V form a separation of $\bigcup_{\alpha\in J} A_\alpha$. Let $x \in \bigcap_{\alpha\in J} A_\alpha$. Suppose $x\in U$.

By lemma 1, since A_α is connected and $x \in A_\alpha$, then $A_\alpha \subseteq U$ $\forall \alpha\in J$

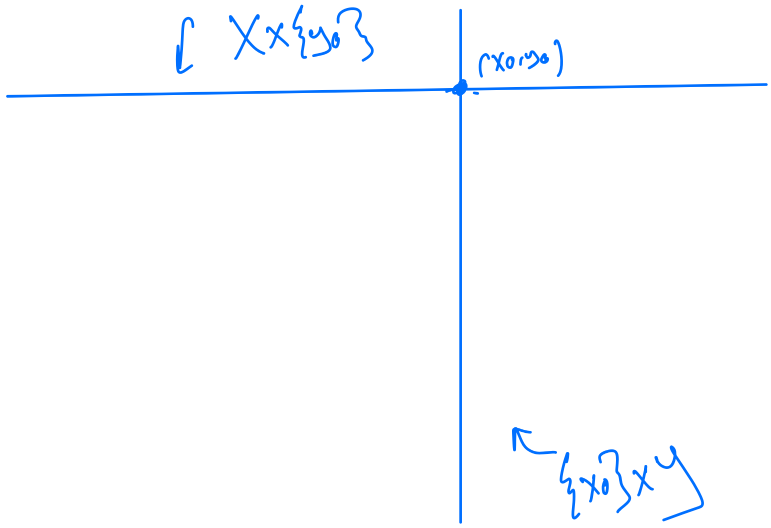
$\Rightarrow \bigcup_{\alpha\in J} A_\alpha \subseteq U \Rightarrow V = \phi$, contradiction

Theorem: A finite product of connected sets is Connected.

Proof: We prove it for a product of 2 spaces X and Y. (Use induction to prove it for any finite product)

For any $(x,y)\in X\times Y$, we know that $\{x\}\times Y$ and $X\times\{y\}$ are homeomorphic to Y and X respectively, and hence connected.

Fix $(x_0,y_0) \in X\times Y$.



Define
$T_{(x_0,y_0)} := X\times\{y_0\} \cup \{x_0\}\cup Y$ which is connected by lemma 2 since they have the point (x_0,y_0) in common.

So $\bigcup_{x\in X} T_{(x,y_0)}$ is connected since each $T_{(x,y_0)}$ is connected for every $x \in X$ and $(x_0,y_0)\in \bigcap_{x\in X} T_{(x,y_0)}$.

Since $X\times Y = \bigcup_{x\in X} T_{(x_0,y_0)}$, we conclude $X\times Y$ is connected.

$\blacksquare$

$\Longrightarrow$ $\mathbb{R}^n$ is connected $\forall n\geq 1$.

Can this be generalized for arbitrary products?

Pathwise connectedness

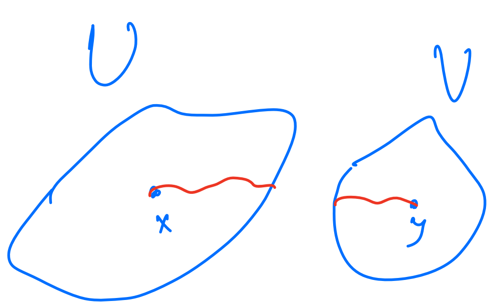
Def: A path in a topological space X from x to y is a continuous function $f: [0,1] \to X$ satisfying $f(0) = x$ and $f(1) = y$.

Def: X is pathwise connected if for every pair $x, y \in X$, $\exists$ path from x to y.

Theorem: Pathwise connected $\Rightarrow$ Connected.

Proof: Suppose U and V form a separation of X. Let $x \in U$ and $y \in V$.
Then $\exists$ path $f: [0,1] \to X$ from x to y.

This implies that $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of $[0,1]$ contradicting the connectedness of $[0,1]$.



Post-lecture - Practice - Questions

1) Do the exercises above.
2) Is $(\mathbb{R}, \mathcal{T}_{co\text{-}finite})$ Connected?
3) a) Let $f: [0,1] \to [0,1]$. Show that $\exists$ a fixed point (i.e. $\exists x \in [0,1]$ s.t. $f(x) = x$)

   b) Show that the fixed-point property described above is a topological invariant.

4) Let p be a point in a connected space X. We say p is a cut-point of X if $X \setminus \{p\}$ is not connected.
Let $f: X \to Y$ be a homeomorphism between connected sets. Show that $f(p)$ is a cut point of Y if p is a cut-point of X.

5) Prove that no two of the intervals $[0,1]$, $[0,1)$, $(0,1)$ are homeomorphic.

   Hint: Use the above two questions.

6) Consider the space $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{box})$. Let $A \subseteq \mathbb{R}^{\mathbb{N}}$ the set of all bounded sequences and $B \subseteq \mathbb{R}^{\mathbb{N}}$ be the

set of all unbounded sequences. Show that A and B form a separation of $(\mathbb{R}^N, \tau_{box})$, and hence it's not connected.

7) Show that the open ball $B(x, r) \subseteq \mathbb{R}^n$ is pathwise connected. Hint: Check example 3 in pg 256.

8) a) Show that continuous functions map pathwise connected sets to pathwise connected sets.

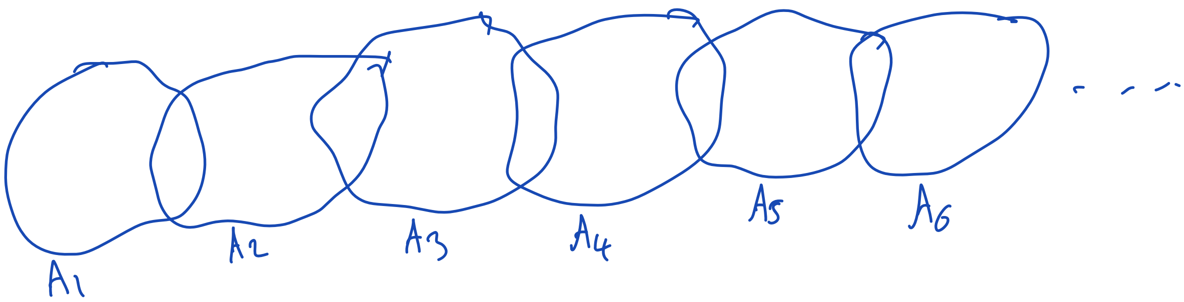
b) Show that S^n is pathwise connected.
Hint: define $g: \mathbb{R}^n \setminus \{0\} \to S^n$ by $g(x) = \frac{x}{\|x\|}$.

9) Define $S = \{ (x, \sin\frac{1}{x}) \mid x > 0 \}$.
a) Find $\bar{S}$.

b) Show that S is connected and therefore $\bar{S}$ is connected.

($\bar{S}$ is called the topologist's sine curve and is an example of a connected space that is not pathwise connected).

10) Let $\{A_n\}_{n \in \mathbb{N}}$ be a collection of connected subspaces of X satisfying $A_n \cap A_{n+1} \neq \phi$. Show $\bigcup_{n \in \mathbb{N}} A_n$ is connected.



11) Let A, B be connected sets in $\overset{\mathbb{R}}{\cancel{X}}$. Show that $A \cap B$ is connected.