* Laithy

1) logistics / info about course

2) Informal Introduction
3) Begin the course content.

* Course website
* Syllabus
* Prerequisites. "Mathematical Maturity"
* Book / Course Content. "Topology" by Munkres
* lecture style
* Tutorials. Arthur Qui and Jiawei Chen
* Help!

  Instructor : 2 weekly office hours (Friday 1-3)

  TAs : 2 weekly office hours

  Piazza :
* Assignments & Academic Integrity

$\underbrace{\text{lectures, tutorials, book, Assignments}}_{\text{will teach you topology}}$

Informal Introduction

Let X be an arbitrary set of things.
no notion of "closeness" or "Position" or "neighborhood"
no notion of "continuous curve/function" on X
no notion of "a sequence x_n converging to x"

But what if $X = \mathbb{R}$? On $\mathbb{R}$, we have:
"distance between x and y" $= d(x,y) = |x-y|$

called a metric.
A set equipped with a metric is called a metric space.
veryyy nice examples of topological spaces

To have the above notions, we need to equip X with a structure that gives it a sense of "place", "locality" or "Position". This is called a topology on X (that type of structure is called a topological structure).

Examples of sets that we equip with a topological structure:

1) * $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ (2-dim sphere)
   (its topology will be inherited from the topology on $\mathbb{R}^3$).
   * Other surfaces in $\mathbb{R}^3$.
   * Manifolds

2) function space (set of functions on X).

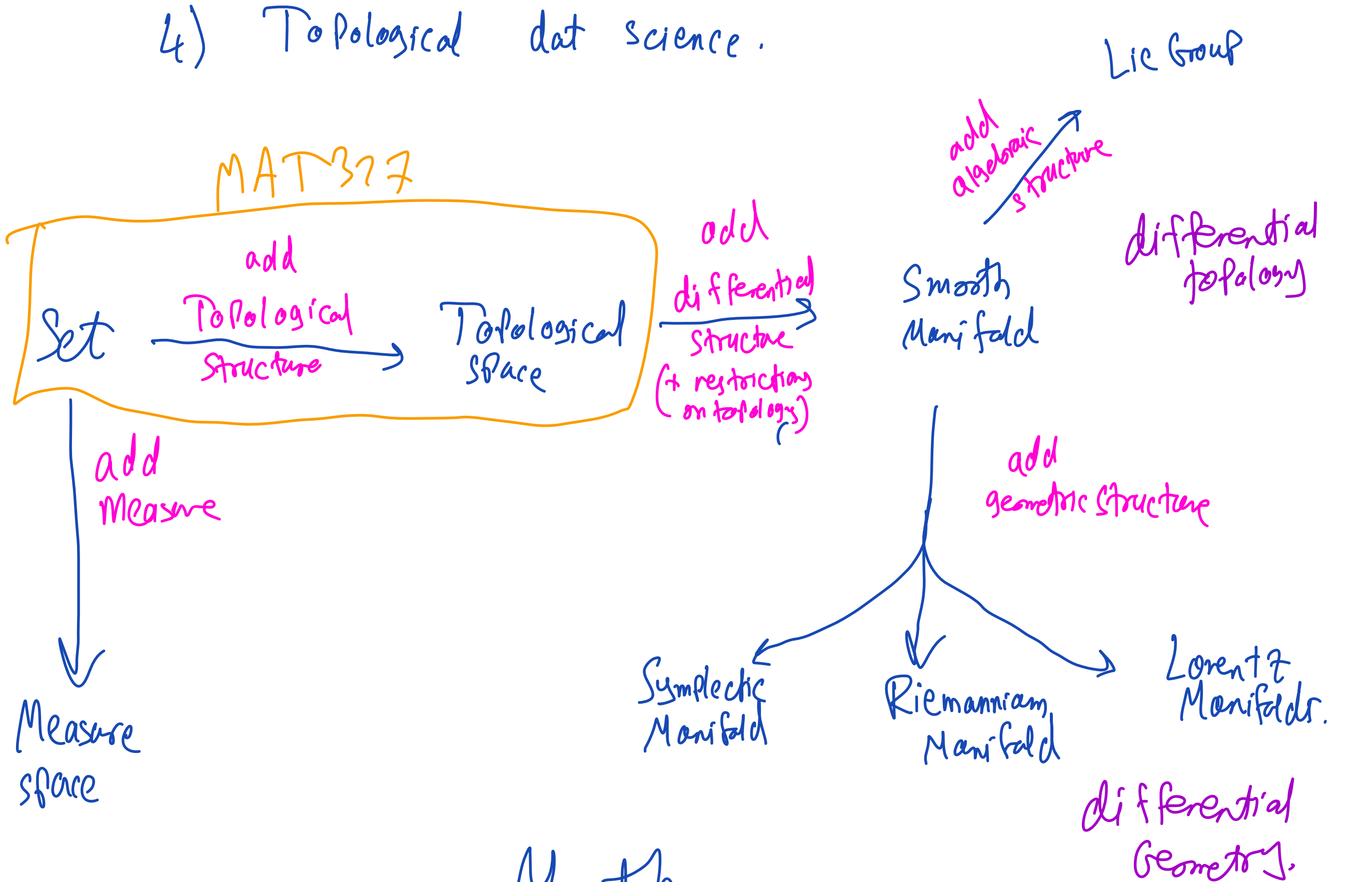
Supremum metric $d_\infty(f,g) = \sup_{x\in X} |f(x)-g(x)|$

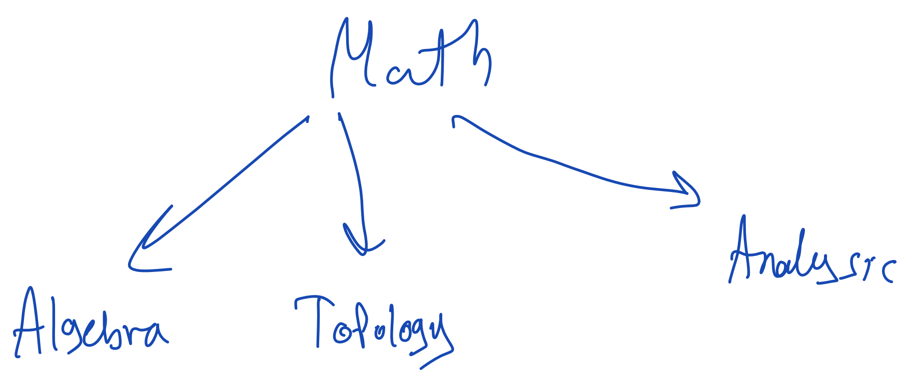
L^p metric $d_p(f,g) = \left(\int |f-g|^p\right)^{1/p}$

give rise to different topologies on the set

3) Set of curves from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$. (where a,b are fixed).

4) Topological dat science.





Definition of Topology

<u>Topology of $\mathbb{R}^n$:</u> ($\mathbb{R}^n$ contains more structure that we need)

The metric on $\mathbb{R}^n$: $d(x,y) := \|x-y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

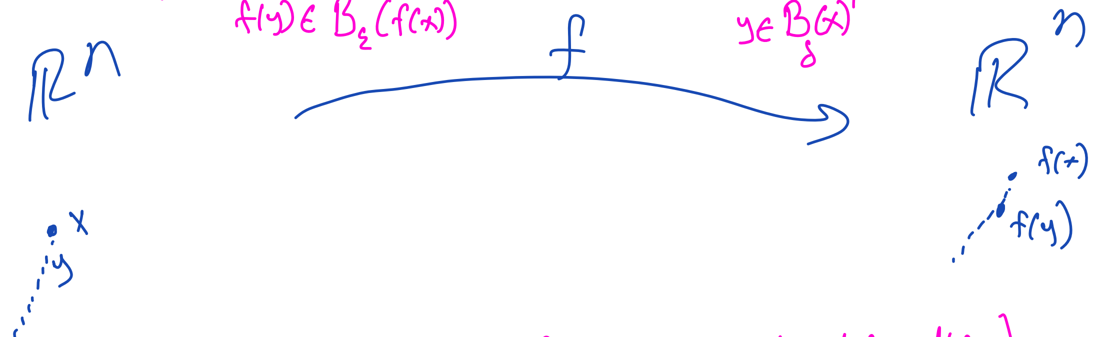
gives us a notion of closeness & continuity.

Recall the def of continuous:

A map $f: \mathbb{R}^n \to \mathbb{R}^n$ is cont if $\forall x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. $\underbrace{d(f(x), f(y)) < \varepsilon}_{f(y) \in B_\varepsilon(f(x))}$ whenever $d(x,y) < \delta$.

$y \in B_\delta(x)$



(also called a contin. deformation)

<u>Def</u>: $f: \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism if f is invertible and continuous with continuous inverse.

A homeomorphism perserves the topological structure.

Topology is the study of all the properties perserved by homeomorphisms or cont. deformations. Those properties are called topological properties (topological invariants).

Question: Is boundedness on $\mathbb{R}^n$ a topological property?

No. Since $f(x) = \tan x$ is a homeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$.

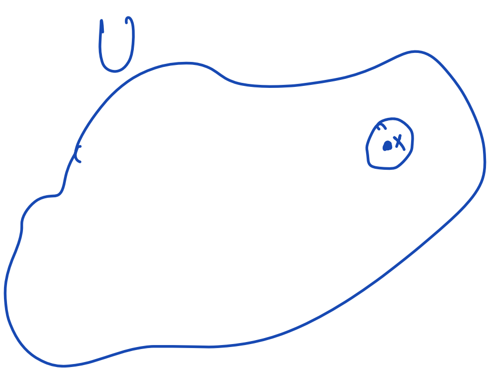
Boundedness comes from the metric, which would not necessarily exist on an arbitrary topological space.

Recall: The open ball $B_r(x)$ centered at x with radius r is defined as the set $\{y \in \mathbb{R}^n \mid d(x,y) < r\}$

Let's define a notion more general than open balls:

Def: $U \subseteq \mathbb{R}^n$ is open if $\forall x \in U$, $\exists r > 0$ s.t. $B_r(x) \subseteq U$

Exc: Show this is equivalent to saying that U is a union of open balls.



We can formulate the notion of closeness and continuity using only open sets.

Exc: $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous iff $f^{-1}(U)$ is open whenever $U \subseteq \mathbb{R}^n$ is open.

This motivates the idea of defining a topological structure by choosing what the open sets of X are.

Is any collection $\mathcal{T} \subseteq P(X)$ can be chosen to be the open sets? No! It will not necessarily give a sensible notion of "closeness". (Since intuitively we want union / intersection of "neighbourhoods" of x to be also "neighbourhoods" of x).

After some thought, we come up with the definition:

Def: A topology on X is a collection $\mathcal{T} \subseteq P(X)$ satisfying:

1) $\emptyset, X \in \mathcal{T}$
2) An arbitrary union of sets in $\mathcal{T}$ is in $\mathcal{T}$.
3) A finite intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

(Sets in $\mathcal{T}$ will be called open sets)

The pair $(X, \mathcal{T})$ is called a topological space.

Why do we restrict (3) to finite intersections?

Ex: Discrete Topology on X is $\tau = P(X)$

(check τ is indeed a topology)

* Every subset is open. In particular $\{x\}$ are open.

singleton

* Every function is cont!!

* sequences only converge if they're eventually constant

* Is this topology a metric space?

Yes! $d(x,y) = \begin{cases} 0, & \text{if } x=y \\ 1, & x \neq y \end{cases}$

show that the topology induced by that metric is indeed the discrete topology.

Ex: Indiscrete Topology on X is $\tau = \{\emptyset, X\}$