Slide 2

also

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

1) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

N^{th} order approximation for e is $\sum_{n=0}^{N} \frac{1}{n!}$

$$e - \sum_{n=0}^{N} \frac{1}{n!}$$

$$= \sum_{n=N+1}^{\infty} \frac{1}{n!} = R_N(1)$$

$$= \frac{e^{\xi} \, 1^{N+1}}{(N+1)!} = \frac{e^{\xi}}{(N+1)!}$$

for some $\xi \in (0, 1)$

Also since $\frac{e^{\xi}}{(N+1)!} \neq 0 \quad \forall N \in \mathbb{N}$,

$$e \neq \sum_{n=0}^{N} \frac{1}{n!},$$

$$e = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} = \sup_{N\in\mathbb{N}} \sum_{n=0}^{N} \frac{1}{n!}$$

2) $$\left| R_N(1) \right| = \frac{e^{\xi}}{(N+1)!} < \frac{e^1}{(N+1)!} < \frac{3}{(N+1)!}$$

so $\forall N \in \mathbb{N}$

$$0 < \left| e - \sum_{n=0}^{N} \frac{1}{n!} \right| < \frac{3}{(N+1)!}$$

suppose $e = \frac{a}{b}$

so we get

$$0 < \left| \frac{a}{b} - \sum_{n=0}^{N} \frac{1}{n!} \right| < \frac{3}{(N+1)!}$$

$\forall N \in \mathbb{N}$

3) find the contradiction:

multiply by $N!$

$$0 < \left| \underbrace{\frac{N!\,a}{b}}_{\text{if } N > b\text{, then this is an integer}} - \underbrace{\sum_{n=0}^{N} \frac{N!}{n!}}_{\text{always an integer}} \right| < \frac{3}{N+1}$$

Choose $N > b$ $\Rightarrow$ so $N > 1$

Then
$$0 < |m_1 - m_2| < \frac{3}{N+1} \leq 1$$

$$\Rightarrow \quad 0 < |m_1 - m_2| < 1$$

where $m_1 = \frac{N!\,a}{b} \in \mathbb{Z}$, $m_2 = \sum_{n=0}^{N} \frac{N!}{n!} \in \mathbb{Z}$

Contradiction

Slide 3

1) $$\lim_{x\to 0} \frac{\overbrace{6\left(\frac{-x^3}{6} + \frac{x^5}{5!} + \cdots\right) + x^3}^{\frac{6x^5}{5!} + \text{h.o.t.}}}{x^5}$$

$$= \frac{6}{5!}$$

3) $$\lim_{x\to 0} \frac{\left(\frac{-x^3}{6} + \cdots\right)^3 x}{\left(x^2/2 + \cdots\right)^4 (x + \cdots)^2}$$

$$= \left(\frac{-1}{6}\right)^3 \Big/ \left(\frac{1}{2}\right)^4$$

$$= \frac{-2}{27}$$

Slide 4

$$\sin x = \sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$2N+1^{th}$ Taylor Polynomial $2N+1$

also $2N+2^{th}$ Taylor Poly.

1)

$$A = \sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$B = \ln(1-x)\Big|_{x=0.1}$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n}\Big|_{x=0.1}$$

$$= -\sum_{n=1}^{\infty} \frac{(0.1)^n}{n}$$

2,3)

$$\left| \sin 1 - \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} \right|$$

$$= \left| R_{2N+1}(1) \right| = \left| \frac{\sin^{(2N+2)}(\xi)\, 1^{2N+2}}{(2N+2)!} \right|$$

$$\leq \frac{1}{(2N+2)!} < 0.001$$

Chose $N=6$

now

$$\left| \ln 0.9 - \sum_{n=0}^{N} \frac{(0.1)^n}{n} \right| = ?$$

$$\left| \frac{-1}{1-x} + \sum_{n=0}^{N} x^n \right| = \left| \frac{1}{1-x} - \sum_{n=0}^{N} x^n \right|$$

$$\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=N+1}^{\infty} x^n = \frac{x^{N+1}}{1-x}$$

true for $|x|<1$

error

Then

$$\left| \int_0^{0.1} \left(\frac{-1}{1-x} + \sum_{n=0}^{N} x^n \right) dx \right| \le \int_0^{0.1} \left| \frac{-1}{1-x} + \sum_{n=0}^{N} x^n \right| dx$$

$$= \int_0^{0.1} \frac{x^{N+1}}{1-x}\, dx$$

$$\le \frac{1}{0.9} \int_0^{0.1} x^{N+1}\, dx$$

$$\left| \ln 0.9 + \sum_{n=0}^{N} \frac{x^n}{n} \right|$$

$$\int_0^{0.1} \frac{-1}{1-x}\,dx = \ln 0.9$$

$$= \frac{1}{0.9} \frac{x^{N+2}}{N+2}\Big|_0^{0.1}$$

$$= \frac{(0.1)^{N+2}}{0.9(N+2)}$$

$$\le \underbrace{\frac{\cancel{10}}{9(N+2)}}_{\to 0 \text{ as } N\to\infty}$$

Chose $N = 1000$

Then $\displaystyle\sum_{n=0}^{1000} \frac{(0.1)^n}{n}$ approximates $\ln 0.9$

with < 0.0001

Slide 5

ln

$$\sin x = \cdots (x-2\pi)(x-\pi)\,x\,(x+\pi)(x+2\pi) \cdots$$

also

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Compare 3^{rd} order

$$-\frac{1}{3!} = \cdots = -\sum_{n=0}^{\infty} \frac{1}{n^2\pi^2}$$

$$\Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

Compare 5^{th} order

$$\frac{1}{5!} = \cdots = \frac{90}{5!} \sum_{n=0}^{\infty} \frac{1}{n^4\pi^2}$$

$$\frac{d}{dx} \ln \sin x = \underline{\cot x}$$

$$=$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$$

$$e^{-2x^2} =$$

Slide 3 :

$\cos 2x$ $\quad \frac{2}{3}x^4 \quad \frac{2^4}{4!} \quad 2^2/3$

2)

$$\lim_{x \to 0} \frac{\left(\cancel{1} - \cancel{\frac{4x^2}{2!}} + \frac{2^4x^4}{4!} + h.o.t\right) - \left[\cancel{1} - \cancel{2x^2} + \frac{4x^4}{2} + \cdots\right]}{x^4}$$

$2x^4 \leftarrow$

$$= \lim_{x \to 0} \frac{-4/3\, x^4 + h.o.t}{x^4} = \frac{-4}{3}$$