

Slide 2

Slide 3

1)

$$f(x) := \sin x = \sin(x-a+a)$$

$$= [\sin(x-a)] \cos a + [\cos(x-a)] \sin a$$

not yet

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^{2n+1}}{(2n+1)!} \right] \cos a + \left[\sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^{2n}}{(2n)!} \right] \sin a$$

method 1: find $f^{(n)}(a)$ $\forall n \in \mathbb{N}$

Then its Taylor series is

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Method 2: Cheat. we know Taylor series of $\sin x$ and $\cos x$ at $x=0$, use that to find Taylor series of $\sin x$ at $x=a$.

we get Taylor series of $\sin x$ about $x=a$ to be:

$$\begin{aligned}
 S(x) &\Rightarrow \cos a \sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^{2n+1}}{(2n+1)!} + \sin a \sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^{2n}}{(2n)!} \\
 &= \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{where} \\
 a_n &= \begin{cases} \frac{(-1)^{n/2}}{n!} \sin a, & n \text{ is even} \\ \frac{(-1)^{(n-1)/2}}{n!} \cos a, & n \text{ is odd} \end{cases}
 \end{aligned}$$

we wts: $\left\{ \begin{array}{l} S(x) = \sin x \text{ on an interval centered} \\ \text{at } a. \end{array} \right\}$

Note: $S(x)$ is defined $\forall x \in \mathbb{R}$ (by The Ratio test)

2) fix $x \in \mathbb{R}$, fix $n \in \mathbb{N}$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \quad \text{for some } c \text{ between } x \text{ and } a.$$

$$3) \quad \forall c \in \mathbb{R}, \quad |f^{(n+1)}(c)| = \left| \begin{matrix} \sin c \\ \text{or} \\ \cos c \end{matrix} \right| \leq 1$$

$$\text{So } |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1} \\ \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

WTS: $\lim_{n \rightarrow \infty} R_n(x)$

$$= \lim_{n \rightarrow \infty} (f(x) - P_n(x))$$

$$= f(x) - \lim_{n \rightarrow \infty} \cancel{P_n(x)}$$

$$= f(x) - S(x)$$

$$= 0$$

On some interval
centered at a .

for any $x \in \mathbb{R}$,

$$4.5) \quad \lim_{n \rightarrow \infty} |R_n(x)| \leq \lim_{n \rightarrow \infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

$$0 \leq$$

$$\Rightarrow \lim_{n \rightarrow \infty} R_n(x) = 0$$

we conclude that $\forall x \in \mathbb{R}$,

$$S(x) = \sin x$$

so $\sin x$ is analytic at a .

$\Rightarrow \sin x$ is analytic on all of \mathbb{R} .

(Always $|x_n| \rightarrow 0 \Rightarrow x_n \rightarrow 0$

(Recall that interval of convergence of any $\sum_{n=0}^{\infty} a_n(x-a)^n$ is all x s.t. the series converges)

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$$A = \int_0^1 x^{17} \sin x \, dx$$

$$= \int_0^1 x^{17} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{2n+18} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+19}}{(2n+1)!(2n+19)} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+19)}$$

clearly

2) find N s.t.

$$\left| A - \sum_{n=0}^N \frac{(-1)^n}{(2n+1)!(2n+19)} \right| < 0.001$$

$$= \left| A - \int_0^1 x^{17} \sum_{n=0}^N \frac{(-1)^n x^{2n+1}}{(2n+1)!} dx \right|$$

$P_{2N+1}(x)$

$$= \left| \int_0^1 x^{17} (\sin x - P_{2N+1}(x)) dx \right|$$

$$\leq \int_0^1 x^{17} |\sin x - P_{2N+1}(x)| dx$$

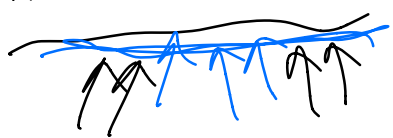
\downarrow

$$|R_{2N+1}(x)| = \left| \frac{\sin^{(2N+2)}(c) x^{2N+2}}{(2N+2)!} \right| \quad c \in (0, x)$$

$$\leq \frac{x^{2N+2}}{(2N+2)!}$$

$$\leq \int_0^1 \frac{x^{17} x^{2N+2}}{(2N+2)!} dx = \frac{x^{2N+20}}{(2N+2)!(2N+20)} \Big|_0^1$$

$$\left| A - \sum_{n=0}^N \frac{(-1)^n}{(2n+1)!} \right| \leq \frac{1}{(2N+2)!(2N+1)}$$



Let $N=1$, Then $\left| A - \frac{1}{3!2!} \right| \leq \frac{1}{24 \cdot 22} < 0.001$

So $A \approx \frac{1}{126} + \frac{1}{19}$ accurate up to an error < 0.01

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$$A = \sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$$

←

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{(2n+1)!} \bigg|_{x=2}$$

←

✓

$$\begin{aligned}
 & \downarrow \\
 & = \frac{1}{\sqrt{x}} \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{x}^{2n+1} (-1)}{(2n+1)!} \\
 & = \frac{1}{\sqrt{x}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{x}^{2n+1}}{(2n+1)!} - 1 + \frac{\sqrt{x}^3}{6} \right] \\
 & = \frac{1}{\sqrt{x}} \left[\sin \sqrt{x} - 1 - \frac{\sqrt{x}^3}{6} \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{So } A &= \frac{1}{\sqrt{x}} \left[\sin \sqrt{x} - 1 - \frac{\sqrt{x}^3}{6} \right] \Big|_{x=2} \\
 &= \frac{1}{\sqrt{2}} \left[\sin \sqrt{2} - 1 - \frac{\sqrt{2}}{3} \right]
 \end{aligned}$$

2)

$$\begin{aligned}
 & x^2 \frac{d}{dx} \sum_{n=0}^{\infty} x^{4n+1} \\
 &= \sum_{n=0}^{\infty} (4n+1) x^{4n+2} = B
 \end{aligned}$$

$$\text{So } B = x^2 \frac{d}{dx} x \sum_{n=0}^{\infty} x^{4n}$$

$$= x^2 \frac{d}{dx} \left[x \sum_{n=0}^{\infty} (x^4)^n \right]$$

$$= x^2 \frac{d}{dx} \left[\frac{x}{1-x^4} \right]$$

$$= x^2 \left[\frac{1}{1-x^4} + \frac{4x^4}{(1-x^4)^2} \right]$$

defined for $|x| < 1$

$$\therefore B(x) =$$

$$3) C = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$$

Reminder:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!} \quad \text{not quite } e^x$$

$$\begin{aligned} e^x + e^{-x} &= 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ &= 2 \sum_{n=0}^{\infty} \frac{(x^2)^n}{(2n)!} \end{aligned}$$

$$\text{so } \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{2} = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$$

$$\text{let } x=2, \quad \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} = \frac{e^{\sqrt{2}} + e^{-\sqrt{2}}}{2}$$

$$4) \quad \int_0^x \phi(t) dt$$

$\frac{1}{2}$

$$\downarrow \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{(2n+1)} \quad \leftarrow 2n! \cdot 2n+1 = 2n+2$$

$$= \frac{2}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n+2}$$

$$= \frac{2}{x} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} - 1 \right]$$

$$= \frac{2}{x} [\cos x - 1]$$

$$\text{So } \int_0^x D(t) dt = \frac{2}{x} (\cos x - 1)$$

$$\text{So } D(x) = \frac{-2(\cos x - 1)}{x^2} - \frac{2}{x} \sin x$$

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1)

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right)$$

$$= \left(1 + x + \frac{x^2}{2} + \dots \right) \left(x - \frac{x^3}{3!} + \dots \right)$$

$$= 1 + x + x^2 + \dots$$

$$+ \frac{1}{6} + \frac{1}{2} x^3 + \dots$$

$$= 1 + x + x^2 + \frac{1}{3} x^3 + \dots$$

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x$$

$$e^{\sin x} = \sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right)^n$$

Compute $g^{(n)}(0)$ until you get

4 non zero terms:

$$g(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

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$$1) \quad g^{(4)}(0) = 4! a_4 = 4! \cdot \frac{-1}{8} = -3$$

$$2) \quad \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1 - x}{x^2 + x^3}$$

$$= \lim_{x \rightarrow 0} \frac{x^2/2 - x^4/8 + \dots}{x^2 + x^3} = \frac{1}{2}$$

3) ---

$$4) \lim_{x \rightarrow 0} \frac{x^2 (e^{\sin x} - 1 - x)}{(\cos x - 1)^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 (x^2/2 - x^4/8 + \dots)}{(-x^2/2 + x^4/4! + \dots)^2}$$

$$= \lim_{x \rightarrow 0} \frac{x^4/2 + \dots}{x^4/4 + \dots} = 2$$

$$\lim_{x \rightarrow 0} \left(\frac{x^2 + x^3}{5x^2 + 10x^4} \right)$$

$$\downarrow \approx \lim_{t \rightarrow 0} \frac{1 + x}{5 + 10x^2} \approx \frac{1}{5}$$