Slide 2

1)

$$g'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{for } |x|<1$$

$$g(x) = \cancel{g(0)}^{\,0} + \int_0^x \frac{1}{1+t}\,dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt \qquad \text{since } |x|<1, \; |t|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n \underbrace{\int_0^x t^n \, dt}_{\frac{x^{n+1}}{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \checkmark$$

$$= \sum_{n=1}^{\infty} \overbrace{\frac{(-1)^{n-1}}{n} x^n}^{a_n x^n} \checkmark$$

2)

$$f'(x) = 2\sin x\cos x$$

$$= \sin \underbrace{2x}_{u}$$

$$= \sin u$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1!} u^{2n+1} \quad \text{for any } u \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2^{2n+1} x^{2n+1} \quad \text{for any } x \in \mathbb{R}$$

$$\left(\sum_{n=0}^{N} a_n = \sum_{n=m}^{N+m} a_{n-m} \right)$$

$$f(x) = \int_0^x f'(t)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} 2^{2n+1} x^{2n+2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{2n!}$$

3) $h'(x) = \boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$ Converges for $|x|<1$

we have shown that $\frac{1}{1+x^2}$ is anal at 0.

$$\underline{h(x)} = \int_0^x h'(t)\,dt$$

$$a_{2n+1} = \frac{(-1)^n}{2n+1} =: \frac{h^{(2n+1)}(0)}{(2n+1)!}$$

$$= \underline{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$$ converges $|x|\leq 1$

Taylor series of arctan. about 0.

Does that mean $h(x) = \arctan x$ is analytic at 0?

Slide 3

1) $x^2 \sum (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

4) $\ln(1+x) - \ln(1-x)$

$\ln(1+(-x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-x)^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} - \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^{n-1} \cdot (-1)^n}^{-1} x^n}{n}$$

$$= \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{2}{n} x^n = \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$$

5) $$(1-x^2) \sum_{n=0}^{\infty} \frac{x^n}{n!} = \cdots$$

2) $$x^5 \left[\sum_{n=6}^{\infty} \frac{(-1)^n}{n} x^{3n} \right] = \cdots$$

3) $$\sin(2x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^3)^{2n+1}}{(2n+1)!} = \cdots$$

6) $$\frac{1}{(1+x^2)(1+x)} = \left(\frac{1}{1+x^2}\right)\left(\frac{1}{1+x}\right)$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right)\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)$$

$$= \frac{1}{2}\left[\frac{1}{1+x} - \frac{x-1}{1+x^2}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n x^n - \underline{(x-1)\sum_{n=0}^{\infty}(-1)^n x^{2n}}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}(-1)^{n+1} x^{2n+1} + \sum_{n=0}^{\infty}(-1)^n x^{2n}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} x^{2n} + \sum_{n=0}^{\infty} -x^{2n+1} + \quad + \quad\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left[1+(-1)^n\right] x^{2n} - \sum_{n=0}^{\infty}\left[1+(-1)^n\right] x^{2n+1}\right]$$

$$= \frac{1}{2}\underline{\sum_{n=0}^{\infty} a_n x^n} \qquad \text{where } a_n = \begin{cases} \left[1+(-1)^{n/2}\right], & n \text{ is even} \\ \left[1+(-1)^{\frac{n-1}{2}}\right], & n \text{ is odd} \end{cases}$$

$$\frac{f^{(n)}(0)}{n!}$$

Problem 2: Recall $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Then $\underbrace{f^{(2020)}(0) = 2020!\, a_{2020}}$

Use This

what about $f^{2020}(1) = ?$

Not easy.

Slide 4

$$f(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt$$

$$= \int_0^x \left\{ \sum_{n=0}^{N} (-1)^n t^{2n} + \sum_{n=N+1}^{\infty} (-1)^n t^{2n} \right\} dt$$

$$= \int_0^x \sum_{n=0}^{N} \cdots + \int \sum_{n=N+1}^{\infty} (-1)^n t^{2n}\, dt$$

$$= \sum_{n=0}^{N} \int_0^x \cdots + \int_0^x \frac{(-1)^{N+1} t^{2N+2}}{1+t^2}\, dt$$

$$= \underbrace{\sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{2n+1}}_{\text{show this}} + \underbrace{\int_0^x \cdots}_{\text{This Remainder}}$$

Taylor Poly P_n $R_n(x)$