Slide 2

1) $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$1 = P(0) = a_0$

$5 = P'(0) = a_1$

$3 = P''(0) = 2a_2 \implies a_2 = \frac{3}{2}$

$-7 = P'''(0) = 6a_3 \implies a_3 = \frac{-7}{6}$

2) $P(x) = 1 + 5x + \frac{3}{2}x^2 + \frac{-7}{6}x^3 + a_4 x^4$

$+ a_5 x^5 + \cdots + a_n x^n$

for any choice of $n \in \mathbb{N}$ and $a_4, \ldots, a_n$

<u>In general:</u> $P(x) = \sum_{k=0}^{n} a_k x^k$ any degree $n$ polynomial,

$P^{(k)}(0) = k!\, a_k$

3) which definition of Taylor Polynomial?
The third one:

$P_n$ is $n^{th}$ Taylor Polynomial if its smallest degree Polynom satisfying $p^{(k)}(0) = f^{k}(0)$ for $0 \leq k \leq n$

$\Updownarrow$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

so $$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k$$

4)

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Slide 3

Problem 1: $f$ is $C^\infty$ at $a$.

Denote Taylor series for $f$ at $a$ by $S$:

$$S(x) := \lim P_n(x) = \sum^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$n \to \infty$ $\qquad n=0 \qquad n!$

Problem 2:

f is anal at a if

$S(x) = f(x)$ for $x$ close enough to a (on an interval centered at a)

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Slide 4

1) $P_1(x) = f(0) + \frac{f'(0)}{1!} \cdot x = 0$

$P_2(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 = 0$

$P_3(x) = \cancel{f(0)} + \cancel{\frac{f'(0)}{1!} x} + \cancel{\frac{f''(0)}{2!} x^2} + \frac{\overbrace{f'''(0)}^{3!}}{3!} x^3$

$= x^3$

$P_n(x) = x^3 \quad \forall n \geq 3$

So The taylor series $S(x) := \lim P_n(x) = x^3 = f(x)$

$n \to \infty$

So $f$ is analytic at 0.

2)

$$f(x) = x^3 = (x-1+1)^3$$
$$= x\left[(x-1)^2 + 2(x-1) + 1\right]$$
$$= (x-1+1)\left[\,---\,\right]$$
$$= (x-1)^3 + 2(x-1)^2 + (x-1)$$
$$+ (x-1)^2 + 2(x-1) + 1$$
$$= (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

$$P_1(x) = 3(x-1) + 1$$

$$P_2(x) = 3(x-1)^2 + 3(x-1) + 1$$

$$P_3(x) = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

$$P_n(x) = P_3(x) \quad \forall n \geq 3$$

So $S(x) := \lim_{n\to\infty} P_n(x) = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$
$= f(x)$

so it is also analytic at 1.

(since $S(x) = f(x)$ for $x$ near $1$, we say $f$ is anal. at $1$)

Another way: $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k$

you will get the same answer.

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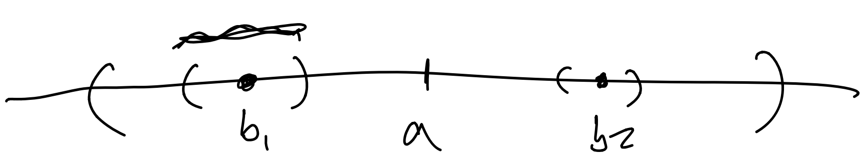
Slide 5

1) Thm:

$$f(x) = S(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{on } I$$

& any Power series is differentiable inf many times in the interior of the interval of convergence



Since $f(x)$ is analytic at any point in $I$, it can be expanded as a power series which we know is differentiable.

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## Slide 7

1) $$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x)^k = 0$$

$\forall n$

So $$S(x) = \lim_{n \to \infty} P_n(x)$$
$$= 0$$

So Taylor series of $f$ at $0$ converges $\forall x$.

But $f$ is not analytic at $0$ since its Taylor series doesn't converge to $f$ near $x=0$. ($S(x) \neq f(x)$ on any interval centered at $0$)

Remainder $R_n(x) := f(x) - P_n(x)$

we know $\lim_{x \to 0} \frac{R_n(x)}{(x-a)^n} = 0$

But $\overbrace{\lim_{n \to \infty} R_n(x)} \neq 0$ on any interval centered at 0,

& so f is not analytic at 0.

Notice: f is analytic at a iff

1) $S(x) = f(x)$ on an interval centered at a.

2) $\lim_{n \to \infty} R_n(x) = 0$ on an interval centered at a.

$f(x) - P_n(x)$

$\sum^{\infty} x^n =: f(x)$ is only defined

$n=0$ on $(-1,1)$

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<u>What can happen:</u> let $f \in C^\infty$

1) Taylor series converges only at $a$.

(f is not analytic at a)

2) Taylor Series Converge on an interval centered at a, but not to f.

(f is not analytic at a)

$S(x) := \sum \frac{f^{(k)}(a)}{k!}(x-a)^k$

3) Taylor series $S(x)$ converges to f on an interval centered at a.

(f is analytic at a)

means $\forall x$ on that interval,

$\sum \cdots (x-a)^k = f(x)$

A Power series is a function of this form:

$$f(x) := \sum_{n=0}^{\infty} a_n (x-a)^n$$

we ask: given the sequence $a_n$, for what values of $x$ does this sum converge.

Answer: 1) only at $x=a$ $\quad \sum_{n=0}^{\infty} n^n x^n$ conv only at $x=0$

2) interval centered at a. $\quad \sum_{n=0}^{\infty} x^n$ converges $\forall x \in (-1,1)$

3) $\mathbb{R}$ $\quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Slide 8

Formula for taylor series:

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

1) $f(x) = e^x = \sum_{k!}^{\infty} \underbrace{\frac{f^{(k)}(2)}{k!}} (x-2)^k$

$$= e^{x-2+2}$$

$$= e^2 e^{(x-2)}$$

$$= e^2 \underbrace{e^u}$$

expand this at $u=0$

$$= e^2 \sum_{n!}^{\infty} \frac{u^n}{n!}$$

$$= e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$$

Converges for all $x$

2) $g(x) = \sin x$

$$= \sin(\underbrace{x - \tfrac{\pi}{4}} + \tfrac{\pi}{4})$$

$$= \frac{1}{\sqrt{2}}\left[\underbrace{\sin(\overbrace{x-\tfrac{\pi}{4}}^{u})} + \underbrace{\cos(\overbrace{x-\tfrac{\pi}{4}}^{u})}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/4)^{2n+1}}{(2n+1)!}} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/4)^{2n}}{2n!}}\right]$$

$$= \ldots$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(x-\pi/4)^n}{n!} \underbrace{(-1)^{\lceil n/2 \rceil}}_{a_n}$$

Converges $\forall x$.

$$a_n = \begin{cases} \ldots \end{cases}$$

3) $$H(x) = \frac{1}{(x-3)+3}$$

We know for any $|r|<1$, $$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$= \frac{1}{3} \frac{1}{1 + \frac{x-3}{3}}$$

$$= \frac{1}{3} \frac{1}{1 - \frac{-(x-3)}{3}} \quad r$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} r^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{3^n}$$

Converges for $|x-3| < 3$

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Slide 10

1) $f'(x) = \frac{1}{1+x^2}$ $\Rightarrow$ $\frac{1}{1+x^2}$ is analytic at $x=0$.

holds for $-1 < x < 1$ $= \sum_{n=0}^{\infty} (-1)^n x^{2n}$ Converges on $|x| < 1$

Recall that inside the interior of the interval of convergence, $\int \sum = \sum \int$

$$f(x) - \cancel{f(0)}^{\,0} = \int_0^x f'(t)\,dt$$

$$f(x) = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \underbrace{\int_0^x t^{2n}\,dt}_{\frac{x^{2n+1}}{2n+1}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{2n+1} + \overbrace{\sum_{n=N+1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}^{R_n}$$

Question: is this the Taylor Polynomial of degree $N$? Yes! Think of why for tomorrow.

(To show $f$ is analytic, we need to show $\lim_{n\to\infty} R_n(x) = 0$ on an interval centered at $0$)