Slide 1:

$$\sum_{n=1}^{\infty} a_n = \sum P.T. + \sum N.T.$$

Note $\rightarrow a_n = \max\{a_n, 0\} + \min\{a_n, 0\}$

If $a_n \geq 0$, $\quad a_n \quad 0 \qquad\qquad 0 \quad a_n$

If $a_n < 0$,

so $S_k = \underbrace{\sum_{n=1}^{k} \max\{a_n, 0\}} + \underbrace{\sum_{n=1}^{k} \min\{a_n, 0\}}$

we know $\sum_{n=1}^{\infty} \max\{a_n, 0\}$, $\sum_{n=1}^{\infty} \min\{a_n, 0\}$ converge.

& so: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \max\{a_n, 0\} + \min\{a_n, 0\}$

$$= \sum_{n=1}^{\infty} \max\{a_n, 0\} + \sum_{n=1}^{\infty} \min\{a_n, 0\}$$

What I used in ⊛ : if $\sum a_n$ conv and $\sum b_n$ conv

used by (1)

Then $\sum a_n + b_n$ conv

& $\sum a_n + b_n = \sum a_n + \sum b_n$

(limit sum Rule).

We also know: If $\sum a_n$ conv, $\sum b_n$ div

$*$ Then $\sum a_n + b_n$ div

(Also limit sum rule)

used by (2) and (3)

(2)

Since $a_n = \max\{a_n, 0\} + \min\{a_n, 0\}$

and $\sum \max\{a_n, 0\} = \infty$

$\sum \min\{a_n, 0\}$ converge

Then by ⊛, $\sum a_n$ div.

$= \infty$

now for (4):

we don't know.

Ex1: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\quad \sum PT = \infty$

$\sum NT = -\infty$

but $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv.

Ex2: $\sum_{n=1}^{\infty} (-1)^n n$ $\quad \sum PT = \infty$

$\sum NT = -\infty$

but $\sum (-1)^n n$ div

since $\lim_{n \to \infty} (-1)^n n \neq 0$

To figure out how it diverges,

Compute S_K.

Slide 5,

1) Converges;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\,\frac{n!}{3^n}$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0$$

So $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges.

2) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!^2\, 3^{n+2}} \cdot \frac{3^{n+1}\, n!}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2\; 3} = \frac{4}{3} > 1$$

so $\sum \dots$ diverges

general version of Ratio Test:

$$\left(\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{n\geq k} a_n \right)$$

instead of limit

3) $\lim_{n\to\infty} \dfrac{1/(n+1)}{1/n} = 1$ so Ratio test is inconclusive.

By integral test, $\sum \frac{1}{n}$ div.

4) $$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1) \cdot n^n}{(n+1)^n \cdot (n+1)} = \lim_{n\to\infty} \frac{\cancel{n+1}}{\cancel{n+1}} \cdot \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} (1+1/n)^{-n}$$

$$= \Big[\underbrace{\lim_{n\to\infty} (1+1/n)^n}_{e}\Big]^{-1}$$

$$\lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+1/n)}$$

$$= \exp\left[\lim_{n\to\infty} n\ln(1+1/n)\right]$$

$$= \exp\left[\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x}\right]$$

$$\overset{L'H'}{=} \exp\left[\lim_{x\to\infty} \frac{\frac{x}{x+1}\cancel{\left(-1/x^2\right)}}{\cancel{-1/x^2}}\right]$$

$$= \exp\left[\lim_{x\to\infty} \frac{x}{x+1}\right] = e$$

So $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1$

So $\sum \frac{n!}{n^n}$ convergent.

5) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = 1$

So Ratio test is inconclusive goes to slower

Try to compare with $1/n$: $\lim\limits_{n\to\infty} \dfrac{1/\ln n}{1/n} = \lim\limits_{n\to\infty} \dfrac{n}{\ln n} = \infty$

goes to 0 faster

so by variation of the LCT,

if $\sum \frac{1}{n}$ div then $\sum \frac{1}{\ln n}$ diverges

(If $\underline{\sum 1/\ln n}$ conv then $\sum \frac{1}{n}$ conv ; useless)

since $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$, we know for n big enough,

$0 < \ln n < n$

so $\dfrac{1}{\ln n} > \dfrac{1}{n} > 0$

by BCT: Then div ... if div

6) $\lim\limits_{n\to\infty} \dfrac{n \ln(n)^2}{(n+1)(\ln(n+1))^2} = \lim\limits_{n\to\infty} \dfrac{n}{n+1} \cdot \left(\lim\limits_{n\to\infty} \dfrac{\ln n}{\ln n+1}\right)^2$

$= 1$

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Ratio test is inconclusive.

Trying with LCT:

$$\lim_{n\to\infty} \frac{\frac{1}{n(\ln n)^2}}{1/n^{0.1}} = \lim \frac{n^{0.1}}{(\ln n)^2} = \infty$$

If $\sum \frac{1}{n^{0.1}}$ div, then $\sum \frac{1}{n(\ln n)^2}$ div

(useless)

Trying with integral test:

since $\frac{1}{n(\ln n)^2}$ is Positive & $\downarrow$

so we can apply integral test.

So The series $\sum \frac{1}{n\ln n^2}$ behaves like

$$\int \frac{1}{x\ln x^2}\,dx$$

So $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{1}{\ln 2} < \infty$

so conv

so $\sum \frac{1}{n \ln n^2}$ also conv.

Slide 6:

Let $\varepsilon > 0$. Since $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$L - \varepsilon \leq \left| \frac{a_{n+1}}{a_n} \right| < L + \varepsilon$$

$$\left[|a_n| (L - \varepsilon) < |a_{n+1}| < |a_n| (L + \varepsilon) \right]$$

$$|a_{n+1}| < |a_n| (L + \varepsilon)$$

$$< |a_{n-1}| (L + \varepsilon)(L + \varepsilon)$$

$$< |a_{n-2}|(L+\varepsilon)$$

$$\vdots$$

$$< |a_N|(L+\varepsilon)^{n-N+1}$$

$$\Rightarrow \quad |a_{n+1}| < |a_N|(L+\varepsilon)^{n-N+1}$$

or $\quad |a_n| < |a_N|(L+\varepsilon)^{n-N}$

holds $\forall n > N$

Similarly apply $|a_n|(L-\varepsilon) < |a_{n+1}|$ several times

to get $\quad |a_N|(L-\varepsilon)^{n-N} < |a_n|$

So $\forall n > N$,

$$\underbrace{|a_N|(L-\varepsilon)^{n-N}}_{\text{geometric series}} < |a_n| < \underbrace{|a_N|(\overbrace{L+\varepsilon}^{r})^{n-N}}$$

(r^n & r^{n-5} are both geometric)

(r^n and $3r^n$ are both geometric)

Problem 2:

suppose $L < 1$.

Choose $\varepsilon = \frac{1-L}{2}$ so $L+\varepsilon < 1$.

$\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|a_n| < (L+\varepsilon)^{n-N}|a_N|$$

Apply BCT: Since $L+\varepsilon < 1$,

$$\sum_{n=N+1}^{\infty} \underbrace{(L+\varepsilon)}_{|r|<1}^{\,n-N} \underbrace{|a_N|}_{\text{constant}}$$ converges

we get $\sum_{n=N+1}^{\infty} |a_n|$ also converges

Then $\sum_{n=1}^{\infty} |a_n|$ converges

Similarly if $L > 1$,

Choose $\varepsilon = \frac{L-1}{2}$ so that $L - \varepsilon > 1$

we know $\exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$0 < (L-\varepsilon)^{n-N} |a_N| < \underline{|a_n|}$

Apply BCT: since $L - \varepsilon > 1$, $\sum_{n=N+1}^{\infty} (L-\varepsilon)^{n-N} |a_N|$ diverges

$|r|>1$ const const

$(L-\varepsilon)^{-N} |a_N| \sum_{n=N+1}^{\infty} (L-\varepsilon)^{n}$

$\to \infty$

so $\sum_{n=N+1}^{\infty} |a_n|$ diverges

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges.

Slide 7:

1) Converges $\forall x \in \mathbb{R}$.

Apply Ratio test: $\lim\limits_{n\to\infty} \left| \dfrac{x^{n+1}}{(n+1)!} \dfrac{n!}{x^n} \right|$

$$= \lim_{n\to\infty} \frac{1}{n+1} |x| = |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0$$

$$\forall x \in \mathbb{R}.$$

So $\forall x \in \mathbb{R}$, $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ converge $= e^x$

2) Apply ratio test.

We want to find all x s.t. $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1} (x-5)^{n+1}}{a_n (x-5)^n} \right| < 1$

$= \left| (x-5) n^2 \quad 4^n \cdot 2 \right|$

$$\lim_{n\to\infty}\left|\frac{[illegible]}{(n+1)^2 \ \& \ 4^{n+1}}\right|$$

$$= \lim_{n\to\infty} |x-5| \frac{n^3}{(n+1)^2} \cdot \frac{1}{4}$$

$$= |x-5| \cdot \frac{1}{4} \underbrace{\lim_{n\to\infty} \frac{n^3}{(n+1)^2}}_{=1}$$

$$= |x-5| \frac{1}{4}$$

1) $\forall x$ s.t. $|x-5|\frac{1}{4} < 1$, $\sum a_n x^n$ converges

$\iff |x-5| < 4$

$\iff x \in (1, 9)$

2) $\forall x$ s.t. $|x-5|\frac{1}{4} > 1$, $\sum a_n x^n$ diverges.

$\iff x \in (9, \infty) \cup (-\infty, 1)$

3) What happens at $x=1$, $x=9$ }

(Ratio test inconclusive).

at $x=1$, $\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{(-4)^n}{2 \cdot 4^n}$

$= \sum_{n=1}^{\infty} \frac{1}{2n^2} (-1)^n$ Converge.

at $x=9$, $\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\cancel{4^n}}{2 \cdot \cancel{4^n}}$

$= \sum_{n=1}^{\infty} \frac{1}{2n^2}$ Conver

Answer is: interval of convergence is $[1, 9]$

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