- Today's lecture will assume you have watched videos 2.7 - 2.11
  For Tuesday's lecture, watch videos 2.12 - 2.13.

Warm up for your first $\varepsilon - \delta$ proof.

1. Find **one** positive value of δ such that
$$|x-3| < \delta \implies |7x-21| < 1.$$

2. Find **all** positive values of δ such that
$$|x-3| < \delta \implies |7x-21| < 1.$$

3. Find **all** positive values of δ such that
$$|x-3| < \delta \implies |7x-21| < \frac{1}{100}.$$

4. Let ε be an arbitrary positive number. Find **all** positive values of δ such that
$$|x-3| < \delta \implies |7x-21| < \varepsilon$$

Your first $\varepsilon - \delta$ proof.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 3} (7x - 15) = 6.$$

Follow these steps, in order:

1. Write down the formal definition of what you're trying to prove. Without the definition, you can't prove anything.
2. Write down what the structure of the proof should be, without filling in any details. What variables must you define in what order, what must you assume and where, etc.
3. Finally, write the complete proof.

A sample proof.

Proof.

$$\begin{aligned} |(7x-15)-6| &< \varepsilon \\ |7x-21| &< \varepsilon \\ 7|x-3| &< \varepsilon \\ |x-3| &< \frac{\varepsilon}{7} \end{aligned}$$

So $\delta = \dfrac{\varepsilon}{7}$. □

(Nearly everything is wrong with this proof.)

This is an example of what the rough work you do *before writing the proof* might look like.

A more complicated $\varepsilon - \delta$ proof.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 0} (x^3 + x^2) = 0.$$

First:

1. Write down the formal definition of what you're trying to prove.
2. Write down what the structure of the proof should be, without filling in any details. What variables must you define in what order, what must you assume and where, etc.
3. Figure out what δ should be given ε (that will be your rough work). Then fill in the details and complete the proof.

A sample proof.

Claim.

$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$

Proof.

Fix $\varepsilon > 0$. Let $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$.

Let $x \in \mathbb{R}$, and assume $0 < |x| < \delta$. Then we have

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

Therefore $|x^3 + x^2| < \varepsilon$, as required. □

A sample proof.

Claim.

$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{such that} \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$

Proof.

Fix $\varepsilon > 0$. Let $\delta = \sqrt{\frac{\varepsilon}{|x+1|}}$.

Let $x \in \mathbb{R}$, and assume $0 < |x| < \delta$. Then we have

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

Therefore $|x^3 + x^2| < \varepsilon$, as required. □

Is the proof correct? If not, what does it do well, and (more importantly) what does it do wrong?

Let's think about choosing δ for this proof.

For all of these questions, let ε and C be fixed positive real numbers.

1. Find **one** value of $\delta > 0$ such that $|x| < \delta \implies Cx^2 < \varepsilon.$
2. Find **all** values of $\delta > 0$ such that $|x| < \delta \implies Cx^2 < \varepsilon.$
3. Find **one** value of $\delta > 0$ such that $|x| < \delta \implies |x+1| < 7.$
4. Find **all** values of $\delta > 0$ such that $|x| < \delta \implies |x+1| < 7.$
5. Find **one** value of $\delta > 0$ such that $|x| < \delta \implies \begin{cases} Cx^2 < \varepsilon \\ |x+1| < 7 \end{cases}.$
6. Find **one** value of $\delta > 0$ such that $|x| < \delta \implies |x^3 + x^2| < \varepsilon.$

Back to the proof.

Problem. Prove, directly from the formal definition of the limit, that

$$\lim_{x \to 0} (x^3 + x^2) = 0.$$

Now you have...

- ...written the formal definition for what you have to prove.
- ...written down the structure of the proof.
- ...figured out how to find a δ that works.

So, now, write the complete proof.

Existence

Write down the formal definition of the following statements:

1. $\lim_{x \to a} f(x) = L$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x)$ does not exist
4. $\lim_{x \to a} f(x) = \infty$

Defining infinite limits

Which of these is a correct definition of $\lim_{x \to a} f(x) = \infty$?

1. $\forall M \in \mathbb{R}$, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.
2. $\forall M > 0$, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.
3. $\forall M > 27$, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.
4. $\forall M \in \mathbb{N}$, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) > M$.
5. $\forall M \in \mathbb{R}$, $\exists \delta > 0$ such that $0 < |x - a| < \delta \implies f(x) \geq M$.

Make sure to think about this with pictures.

Checking your understanding of these conditionals

Let $a \in \mathbb{R}$. Let f be a function. Assume we know the following is true about f:

$$0 < |x - a| < \frac{1}{5} \implies f(x) > 70.$$

Take a minute to draw a picture of what it means for f to satisfy this conditional. It's **much** easier to think about these things when you can visualize them.

1. Which positive values of δ do you know *must* satisfy

$$0 < |x - a| < \delta \implies f(x) > 70?$$

2. Which values of M do you know *must* satisfy

$$0 < |x - a| < \frac{1}{5} \implies f(x) > M?$$

What is the limit of this function?

Let f, g and h be functions defined on $\mathbb{R} \setminus \{0\}$ as:

$$f(x) = \frac{1}{x^2}$$

$$g(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ where } n \in \mathbb{N} \text{ and } n < 10^{100} \\ \frac{1}{x^2} & \textit{otherwise} \end{cases}$$

$$h(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ \frac{1}{x^2} & \textit{otherwise} \end{cases}$$

Evaluate with proof:

1. $\lim_{x \to 0} f(x)$
2. $\lim_{x \to 0} g(x)$
3. $\lim_{x \to 0} h(x)$